MA563 Calculus of Variations Final Assignment

Questions 1 and 2 are worth 40 marks and count for Assignment 3 in SDS. Questions 3 and 4 are worth 40 marks and count for Assignment 4 in SDS. Together, they are worth 10% of the final mark.

DATE DUE: noon, April 9, 2010.

Q1. Show that

$$\begin{pmatrix} \alpha \cdot x \\ \alpha \cdot y \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

is a one-parameter group action.
Suppose further that $x = x(t)$, $y = y(t)$ and $\alpha \cdot t = t$. Show that $\sigma = x^2 - y^2$ and $\tau = x\dot{y} - y\dot{x}$ are invariants.
Show that $\dot{x}^2 - \dot{y}^2$ can be expressed as a function of σ, $\dot{\sigma}$ and τ. What is the general form of a first order Lagrangian invariant under the action (1)?
[10 marks]

Q2. Suppose that the variational problem

$$\mathcal{L}[u, v] = \int L(x, u, u_x, u_{xx}) \, \mathrm{d}x \tag{2}$$

is invariant under a one-parameter group action $u \mapsto \alpha \cdot u$ that leaves the independent variable x invariant. Define what is meant by the *infinitesimals* ϕ, $\phi_{[x]}$ and $\phi_{[xx]}$. Show that

$$\phi_{[x]} = \frac{\mathrm{d}}{\mathrm{d}x}\phi, \qquad \phi_{[xx]} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi$$

where $\mathrm{d}/\mathrm{d}x$ is the total derivative with respect to x.

[6 marks]

Show that

$$0 = \phi E(L) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial u_x}\phi + \frac{\partial L}{\partial u_{xx}} \frac{\mathrm{d}}{\mathrm{d}x}\phi - \phi \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial L}{\partial u_{xx}} \right).$$

[10 marks]

Suppose now that the action is

$$\alpha \cdot u = \exp(2\alpha) u + \frac{1}{2} x \left(\exp(2\alpha) - 1\right).$$

Show that both

$$\frac{2u_x + 1}{2u + x}, \qquad \frac{u_{xx}}{2u + x}$$

are invariant. If

$$L = L\left(\frac{2u_x + 1}{2u + x}, \frac{u_{xx}}{2u + x} \right)$$

show that a first integral of $E(L) = 0$ is

$$c = 2D_1(L) + 2\left(\frac{2u_x + 1}{2u + x}\right) D_2(L) - \frac{\mathrm{d}}{\mathrm{d}x} D_2(L)$$

where c is a constant and $D_i(L)$ is the derivative of L with respect to its ith argument.

[14 marks]

Q3. Consider the Hamiltonian $H(x, p) = kxp$ where k is a constant. Show Hamilton's equations are

$$\dot{x} = kx, \qquad \dot{p} = -kp.$$

Calculate $X = x(t)$, $P = p(t)$ with initial data $x(0) = x$, $p(0) = p$.
Show that the time t flow mapping $(x, p) \mapsto (X, P)$ is symplectic.

[10 marks]

Q4. Consider the Hamiltonian

$$H = \frac{1}{2}\left(x_1^2 + x_2^2 + p_1^2 + p_2^2\right) + x_1 p_2 - x_2 p_1. \tag{3}$$

Find and solve Hamilton's equations for this Hamiltonian.

[10 marks]

Show that the one-parameter group action

$$\alpha \cdot \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} \tag{4}$$

leaves the Hamiltonian (3) invariant. Show further that for each α, the map (4) is a symplectic transformation.

[10 marks]

Show that the one-parameter group action is the time=α flow map of the Hamiltonian system associated with $C = x_1 p_2 - x_2 p_1$. Show that the time t flow map for the Hamiltonian H *commutes* with the one-parameter group action, that is, if you flow for time t and then act by α, the result is the same as if you act by α and then flow for time t.
Verify that C is a conserved quantity for the Hamiltonian flow for H, that is, the Poisson bracket $\{C, H\} = 0$. Is H a conserved quantity for the Hamiltonian flow for C^2?

[10 marks]