MA563 Calculus of Variations
Assignment 4

(Q3 on Sheet) $H(x,p) = kxp \qquad k \in \mathbb{F}$.

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p} = kx$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kp$$

Solving these yields

$$X = x(t) = x(0)e^{kt} = xe^{kt}$$

$$P = p(t) = p(0)e^{-kt} = pe^{-kt}$$

The map $(x,p) \mapsto (X,P)$ is symplectic if

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\partial(X,P)}{\partial(x,p)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial(X,P)}{\partial(x,p)}^T$$

~~So~~ Now

$$\frac{\partial(X,P)}{\partial(x,p)} = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial p} \\ \frac{\partial P}{\partial x} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{pmatrix}$$

and $\begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{pmatrix}$

$$= \begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{pmatrix} \begin{pmatrix} 0 & e^{-kt} \\ -e^{kt} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ as req'd.

(Q4 on sheet)

If $H = \frac{1}{2}\left(x_1^2 + x_2^2 + p_1^2 + p_2^2\right) + x_1 p_2 - x_2 p_1$

then Hamilton's eqns are

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = p_1 - x_2 \qquad — ①$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = p_2 + x_1 \qquad — ②$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -x_1 + p_2 \qquad — ③$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -x_2 + p_1 \qquad — ④$$

Thus
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}}_{Z} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}$$

The soln is
$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} = e^{tZ} \begin{pmatrix} x_1(0) \\ x_2(0) \\ p_1(0) \\ p_2(0) \end{pmatrix}$$

Using Maple,

$$Z := \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

```
> LinearAlgebra[MatrixExponential](t*Z):    simplify(%,trig);
```

$$e^{tZ} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\cos(2\,t) & -\frac{1}{2}\sin(2\,t) & \frac{1}{2}\sin(2\,t) & \frac{1}{2}\cos(2\,t) - \\ \frac{1}{2}\sin(2\,t) & \frac{1}{2} + \frac{1}{2}\cos(2\,t) & -\frac{1}{2}\cos(2\,t) + \frac{1}{2} & \frac{1}{2}\sin(2 \\ -\frac{1}{2}\sin(2\,t) & -\frac{1}{2}\cos(2\,t) + \frac{1}{2} & \frac{1}{2} + \frac{1}{2}\cos(2\,t) & -\frac{1}{2}\sin \\ \frac{1}{2}\cos(2\,t) - \frac{1}{2} & -\frac{1}{2}\sin(2\,t) & \frac{1}{2}\sin(2\,t) & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

To show the one parameter group action leaves the Hamiltonian invariant, we calculate

$$\alpha \cdot H = \frac{1}{2}\left[(\alpha \cdot x_1)^2 + (\alpha \cdot X_2)^2 + (\alpha \cdot p_1)^2 + (\alpha \cdot p_2)^2\right]$$

$$+ (\alpha \cdot x_1)(\alpha \cdot p_2) - (\alpha \cdot x_2)(\alpha \cdot p_1)$$

expand the RHS and show that α disappears, once you apply $\cos^2\alpha + \sin^2\alpha = 1$.

In Maple this looks like

```
> H:=1/2*(x1^2+x2^2+p1^2+p2^2)+x1*p2-x2*p1;
```

$$H := \frac{1}{2}\,x1^2 + \frac{1}{2}\,x2^2 + \frac{1}{2}\,p1^2 + \frac{1}{2}\,p2^2 + x1\,p2 - x2\,p1$$

```
> subs(x1=cos(a)*X1-sin(a)*X2,x2=sin(a)*X1+cos(a)*X2,p1=cos(a)*P1-sin(a)*P2,
p2=sin(a)*P1+cos(a)*P2,H):simplify(%,trig);
```

$$\frac{1}{2}\,X2^2 + \frac{1}{2}\,X1^2 + \frac{1}{2}\,P2^2 + \frac{1}{2}\,P1^2 - X2\,P1 + X1\,P2$$

To show the map (4) is symplectic, where

$$\begin{pmatrix} X \\ P \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ p \end{pmatrix} = A(\alpha) \begin{pmatrix} X \\ P \end{pmatrix} \qquad (4)$$

We need to show

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} = \frac{\partial(\underline{X},\underline{P})}{\partial(\underline{x},\underline{p})} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \frac{\partial(\underline{X},\underline{P})}{\partial(\underline{x},\underline{p})}^T$$

where $\dfrac{\partial(\underline{X},\underline{P})}{\partial(\underline{x},\underline{p})} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial p_2} \\ \vdots & & \vdots \\ \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_2}{\partial p_2} \end{pmatrix}$

Now, the map (4) is linear & so

$$\frac{\partial(\underline{X},\underline{P})}{\partial(\underline{x},\underline{p})} = A(\alpha)$$

So need to show

$$A(\alpha) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} A(\alpha)^T = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

which can be done by noting

$$A(\alpha) = \begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\alpha) \end{pmatrix}$$

where $R(\alpha)$ = rotation matrix (2×2) with angle equal to α

& $R(\alpha)^T = R(\alpha)^{-1}$

& so

$$\begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\alpha) \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} R(\alpha)^T & 0 \\ 0 & R(\alpha)^T \end{pmatrix}$$

$$= \begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\alpha) \end{pmatrix} \begin{pmatrix} 0 & R(\alpha)^{-1} \\ -R(\alpha)^{-1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

as req'd.

Can also use Maple & also can <u>just</u> do the calculation by hand !!

Next, we need to obtain the Hamiltonian system with

$$G = x_1 p_2 - x_2 p_1$$

as the Hamiltonian. Thus we solve

$$\left.\begin{aligned} \dot{x}_1 &= \frac{\partial G}{\partial p_1} = -x_2 \\ \dot{x}_2 &= \frac{\partial G}{\partial p_2} = x_1 \end{aligned}\right\} \quad (*)$$

$$\left.\begin{aligned} \dot{p}_1 &= -\frac{\partial G}{\partial x_1} = -p_2 \\ \dot{p}_2 &= -\frac{\partial G}{\partial x_2} = p_1 \end{aligned}\right\} \quad (**)$$

These are easy to solve: we have

$$x_1(t) = \cos t . x_1(0) - \sin(t) x_2(0)$$

$$x_2(t) = \sin t . x_1(0) + \cos(t) x_2(0)$$

$$p_1(t) = \cos t \ p_1(0) - \sin t \ p_2(0)$$

$$p_2(t) = \sin t \ p_1(0) + \cos t \ p_2(0)$$

or

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ p_1(0) \\ p_2(0) \end{pmatrix} \qquad (***)$$

Comparing this with (4) we see the group action is the same as (***) with $t = \alpha$ as required.

To show the group action commutes with the Hamiltonian flow for H, we have since both maps are linear only to show that

$$e^{tZ} A(\alpha) = A(\alpha) e^{tZ}$$

This is best done using Maple!

Finally, we show $\{\mathcal{C}, H\} = 0$

$$\{\mathcal{C}, H\} = \frac{\partial \mathcal{C}}{\partial x_1}\frac{\partial H}{\partial p_1} + \frac{\partial \mathcal{C}}{\partial x_2}\frac{\partial H}{\partial p_2}$$

$$- \frac{\partial \mathcal{C}}{\partial p_1}\frac{\partial H}{\partial x_1} - \frac{\partial \mathcal{C}}{\partial p_2}\frac{\partial H}{\partial x_2}$$

$$= p_2(p_1 - x_2) + (-p_1)(p_2 + x_1)$$
$$- (-x_2)(x_1 + p_2) - (x_1)(x_2 - p_1)$$
$$= 0.$$

Finally, is H a conserved quantity for $\mathcal{C}^2$? The answer is yes

since $\{H, \mathcal{C}^2\} = 0$

indeed, $\{H, \mathcal{C}^2\} = -2\mathcal{C}\{\mathcal{C}, H\} = 0.$