MA563 Calculus of Variations

Assignment 3 Solutions

Q1 (i) $0 \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

So $\alpha = 0$ is the identity

(ii) $$\beta \cdot \left(\alpha \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right) = \beta \cdot \left(\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= A(\alpha) \begin{pmatrix} \beta \cdot x \\ \beta \cdot y \end{pmatrix} = A(\alpha)\, A(\beta) \begin{pmatrix} x \\ y \end{pmatrix}$$

~~so need to sho~~ where

$$A(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

so need to show

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}\begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} = \begin{pmatrix} \cosh(\alpha+\beta) & \sinh(\alpha+\beta) \\ \sinh(\alpha+\beta) & \cosh(\alpha+\beta) \end{pmatrix}$$

But $$\text{LHS} = \begin{pmatrix} \cosh\alpha\cosh\beta + \sinh\alpha\sin\beta & \cosh\alpha\sinh\beta + \sinh\alpha\cosh\beta \\ \cosh\beta\sinh\alpha + \sinh\beta\cosh\alpha & \sinh\alpha\sinh\beta + \cosh\alpha\cosh\beta \end{pmatrix}$$

$= \text{RHS}$ by well known identities.

Since $\alpha \cdot t = t$, we have

$$\alpha \cdot \dot{x} = \frac{d}{dt} \alpha \cdot x \quad , \quad \alpha \cdot \dot{y} = \frac{d}{dt} \alpha \cdot y$$

or $$\alpha \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(\alpha) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Thus

$$\alpha \cdot (x^2 - y^2) = (\alpha \cdot x)^2 - (\alpha \cdot y)^2$$

$$= (\cosh\alpha\, x + \sinh\alpha\, y)^2 - (\sinh\alpha\, x + \cosh\alpha\, y)^2$$

$=$ (after simplification)

$$(\cosh\alpha^2 - \sinh\alpha^2)(x^2 - y^2)$$

$$= x^2 - y^2$$

and

$$\alpha \cdot (x\dot{y} - y\dot{x}) = (\alpha \cdot x)(\alpha \cdot \dot{y}) - (\alpha \cdot y)(\alpha \cdot \dot{x})$$

$$= (\cosh\alpha\, x + \sinh\alpha\, y)(\sinh\alpha\, \dot{x} + \cosh\alpha\, \dot{y})$$

$$- (\sinh\alpha\, x + \cosh\alpha\, y)(\cosh\alpha\, \dot{x} + \sinh\alpha\, \dot{y})$$

$$= (\cosh\alpha^2 - \sinh\alpha^2)(x\dot{y} - y\dot{x})$$

all other terms cancelling

$$= x\dot{y} - y\dot{x}$$

To show $\dot{x}^2 - \dot{y}^2$ is a function of
$\dot{\sigma} = 2x\dot{x} - 2y\dot{y}$ & $\tau = x\dot{y} - y\dot{x}$

$$\dot{\sigma}^2 = 4x^2\dot{x}^2 + 4y^2\dot{y}^2 - 8x\dot{x}y\dot{y}$$

$$\tau^2 = x^2\dot{y}^2 + y^2\dot{x}^2 - 2x\dot{x}y\dot{y}$$

so $$\dot{\sigma}^2 - 4\tau^2 = 4(x^2 - y^2)(\dot{x}^2 - \dot{y}^2)$$

i.e. $$\dot{x}^2 - \dot{y}^2 = \frac{\dot{\sigma}^2 - 4\tau^2}{4\sigma}$$

We have a one parameter group acting on $(t, x, y, \dot{x}, \dot{y})$ space which has dimension 5 so expect $4 = 5-1$ independent invariants. These are

$$t, \sigma, \dot{\sigma} \text{ and } \tau$$

~~so~~ since $\dot{\sigma}$ is invariant (can be shown directly or by noting it is the derivative of an invariant wrt an invariant indept variable) thus the general form of an invariant Lagrangian is

$$L\,dt = L(t, \sigma, \dot{\sigma}, \tau)\,dt.$$

noting t is the independent variable.

Q2 We have $\alpha \cdot x = x$

and $\alpha \cdot u$, ~~$\alpha \cdot u_x$~~ is not yet given.

Now $\phi = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u$

$$\phi_{[x]} = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u_x$$

$$\phi_{[xx]} = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u_{xx}$$

is the definition of the infinitesimals.
Since $\alpha \cdot x = x$, we have

$$\alpha \cdot u_x = \frac{d(\alpha \cdot u)}{dx} \Big/ \frac{d(\alpha \cdot x)}{dx} = \frac{d}{dx}(\alpha \cdot u)$$

& then $\frac{d}{d\alpha} \frac{d}{dx} (\alpha \cdot u) = \frac{d}{dx}\left(\frac{d}{d\alpha} \alpha \cdot u\right)$

so that $\phi_{[x]} = \left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u_x = \frac{d}{dx}\left(\left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u\right)$

$$= \frac{d}{dx} \phi$$

Similarly, $\alpha \cdot u_{xx} = \frac{d^2}{dx^2} \alpha \cdot u$

& thus $\phi_{[xx]} = \frac{d^2}{dx^2} \left.\frac{d}{d\alpha}\right|_{\alpha=0} \alpha \cdot u = \frac{d^2}{dx^2} \phi$

We are given that the Lagrangian is invariant & that $\alpha \cdot x = x$.

Hence $\alpha \cdot dx = dx$ and

$$\alpha \cdot \left(L(x, u, u_x, u_{xx}) dx \right)$$

$$\left.\begin{aligned} &= L(x, \alpha \cdot u, \alpha \cdot u_x, \alpha \cdot u_{xx}) \, dx \\ &= L(x, u, u_x, u_{xx}) \, dx \end{aligned}\right\} \quad (*)$$

for all α. Differentiating $(*)$ wrt α and setting $\alpha = 0$ yields

$$\frac{\partial L}{\partial u} \phi + \frac{\partial L}{\partial u_x} \phi_{[x]} + \frac{\partial L}{\partial u_{xx}} \phi_{[xx]} = 0 \quad (**)$$

Substituting in that $\phi_{[x]} = \frac{d\phi}{dx}$, $\phi_{[xx]} = \frac{d^2\phi}{dx^2}$

we have

$$0 = \frac{\partial L}{\partial u} \phi + \frac{\partial L}{\partial u_x} \frac{d\phi}{dx} + \frac{\partial L}{\partial u_{xx}} \frac{d^2\phi}{dx^2} \quad (***)$$

Eqn $(***)$ must be the starting point to answer the Q. There are two ways to proceed to show the identity in the question. The first is to expand out what is given in the Q & show it equals $(***)$. This requires knowing that for a second order Lagrangian,

$$E(L) = \frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial L}{\partial u_{xx}}$$

The other method is to use "integration by parts" twice on (***). This proceeds as follows.

$$0 = \frac{\partial L}{\partial u}\phi + \frac{\partial L}{\partial u_x}\frac{d\phi}{dx} + \frac{\partial L}{\partial u_{xx}}\frac{d^2\phi}{dx^2}$$

$$= \frac{\partial L}{\partial u}\phi - \frac{d}{dx}\frac{\partial L}{\partial u_x}\phi + \frac{d}{dx}\left(\frac{\partial L}{\partial u_x}\phi\right)$$

$$- \frac{d}{dx}\frac{\partial L}{\partial u_{xx}}\frac{d\phi}{dx} + \frac{d}{dx}\left(\frac{\partial L}{\partial u_{xx}}\frac{d\phi}{dx}\right)$$

$$= \frac{\partial L}{\partial u}\phi - \frac{d}{dx}\frac{\partial L}{\partial u_x}\phi + \frac{d}{dx}\left(\frac{\partial L}{\partial u_x}\phi + \frac{\partial L}{\partial u_{xx}}\frac{d\phi}{dx}\right)$$

$$+ \frac{d^2}{dx^2}\frac{\partial L}{\partial u_{xx}}\phi - \frac{d}{dx}\left(\frac{d}{dx}\frac{\partial L}{\partial u_{xx}}\phi\right)$$

$$= E(L)\,\phi$$

$$+ \frac{d}{dx}\left(\frac{\partial L}{\partial u_x}\phi + \frac{\partial L}{\partial u_{xx}}\frac{d\phi}{dx} - \frac{d}{dx}\frac{\partial L}{\partial u_{xx}}\phi\right)$$

collecting terms & noting that for a second order Lagrangian,

$$E(L) = \frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial L}{\partial u_{xx}}$$

If $\alpha \cdot u = e^{2\alpha} u + \frac{1}{2} \cdot x \left(e^{2\alpha} - 1\right)$

then $\alpha \cdot u_x = \frac{d(\alpha \cdot u)}{dx} \Big/ \frac{d(\alpha \cdot x)}{dx}$

$= \frac{d(\alpha \cdot u)}{dx}$ as $\alpha \cdot x = x$

$= e^{2\alpha} u_x + \frac{1}{2}\left(e^{2\alpha} - 1\right)$

and $\alpha \cdot u_{xx} = e^{2\alpha} u_{xx}$

and so $\phi = \frac{d}{d\alpha}\Big|_{\alpha=0} \alpha \cdot u = 2u + x$.

~~x~~ Next, we have

$$\alpha \cdot \left(\frac{2u_x + 1}{2u + x}\right) = \frac{2(\alpha \cdot u_x) + 1}{2(\alpha \cdot u) + \alpha \cdot x}$$

$$= \frac{2\left(e^{2\alpha} u_x + \frac{1}{2}\left(e^{2\alpha} - 1\right)\right) + 1}{2e^{2\alpha} u + x\left(e^{2\alpha} - 1\right) + x}$$

$\cancel{= 2\left(e^{2\alpha} u_x + \frac{1}{2}\right(}$

$$= \frac{e^{2\alpha}(2u_x + 1)}{e^{2\alpha}(2u + x)} = \frac{2u_x + 1}{2u + x}$$

and $\alpha \cdot \left(\dfrac{u_{xx}}{2u+x}\right) = \dfrac{e^{2\alpha} u_{xx}}{2\left[e^{2\alpha}u + \frac{1}{2}x(e^{2\alpha}-1)\right] + x}$

$$= \frac{e^{2\alpha} u_{xx}}{e^{2\alpha}(2u+x)}$$

$$= \frac{u_{xx}}{2u+x}$$

& so both expressions are invariant.

Finally, if $L = L\left(\dfrac{2u_x+1}{2u+X}, \dfrac{u_{xx}}{2u+x}\right)$ then $L\,dx$

is invariant & so on solutions of $E(L)=0$ we have by the chain rule

$$C = \frac{\partial L}{\partial u_x}\phi + \frac{\partial L}{\partial u_{xx}}\frac{d\phi}{dx} - \frac{d}{dx}\frac{\partial L}{\partial u_{xx}} \cdot \phi$$

$$= \left(D_1(L)\frac{2}{2u+x}\right) \cdot (2u+x)$$

$$+ \left(D_2(L)\frac{1}{2u+x}\right)\frac{d}{dx}(2u+x)$$

$$- \frac{d}{dx}\left[D_2(L)\frac{1}{2u+x}\right](2u+x)$$

$$= D_1(L)\cdot 2 + D_2(L)\left(\frac{2u_x+1}{2u+x}\right)$$

$$- \frac{d}{dx}D_2(L) - D_2(L)\cdot\left(\frac{-1}{(2u+x)^2}\cdot(2u_x+1)\right)\cdot(2u+x)$$

$$= 2D_1(L) - \frac{d}{dx}D_2(L)$$

$$+ D_2(L)\left[\frac{2u_x+1}{2u+x} + \frac{2u_x+1}{2u+x}\right]$$

$$= 2D_1(L) + 2D_2(L)\cdot\frac{2u_x+1}{2u+x} - \frac{d}{dx}D_2(L)$$

as req'd.