Mathematical Analysis (MA552)

Exercise Sheet Three

1. Use the (ε, δ)-definition of continuity to show that the following functions are continuous at $x = 1$.
   (a) $f(x) = x^5 + 2x - 1$.
   (b) $g(x) = \sqrt{x+1}$.
   (c) $h(x) = 1/x$.

2. Use the (ε, δ)-definition of continuity to show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is continuous.

   Hint: See the proof for $f(x) = x^2$ from the lecture.

3. Determine the following limits:

$$\text{(a)} \quad \lim_{x\to 1} \quad \log\left(x^2 + \frac{\sin(x-1)}{x-1}\right).$$

$$\text{(b)} \quad \lim_{x\to e^\pi} \quad x \cdot \cos(\log x).$$

   Justify your answers.

4. Give examples of functions with the following properties:
   (a) $f : (1, 2) \to \mathbb{R}$ is continuous and unbounded
   (b) $g : [1, 2] \to \mathbb{R}$ is not continuous and unbounded
   (c) $h : (0, 1) \to \mathbb{R}$ is continuous and bounded but does not attain its bounds
   (d) $k : \mathbb{R} \to \mathbb{R}$ is continuous and bounded but does not attain its bounds

5. (a) Classify the following subsets of $\mathbb{R}$ as bounded or unbounded. If the set is bounded, write down the supremum and infimum.

$$\text{(i) } \{x : |x+1| \le 2\}; \quad \text{(ii) } \{1/n^2 : n \in \mathbb{N} \text{ and } n \text{ is prime}\};$$

$$\text{(iii) } \{x^2 : x \in \mathbb{R}\}; \quad \text{(iv) } \bigcap_{n=1}^{\infty}(-1/n, 1/n).$$

   (There is no need to provide proofs of your assertions.)

   (b) Prove that $\sup(1, 2) = 2$.
   Hints: (i) Show that 2 is an upper bound.
   (ii) Assume that you have a smaller upper bound than 2 and derive a contradiction.

6. (a) Show that the equation $x^7 - x^3 + x^2 + 1 = 0$ has at least one real solution.
   (b) Show that there exists $c \in [0, 1]$ such that $\sin(\cos(c)) = c$.

7. Complete the proof of Theorem 2 by showing the following by contradiction:
   If for every sequence (x_n) such that $x_n \to a$ we have $f(x_n) \to f(a)$ as $n \to \infty$, then f is continuous at a.

   Hints: (i) Explain that "f is not continuous at a" means that there exists some $\varepsilon > 0$ such that for every $\delta > 0$, we can find an x satisfying $|x - a| < \delta$ and $|f(x) - f(a)| \ge \varepsilon$.
   (ii) Suppose f is not continuous at a and use (i) to find a sequence (x_n) with $|x_n - a| < 1/n$ and $|f(x_n) - f(a)| \ge \varepsilon$ for some $\varepsilon > 0$. Derive a contradiction.

8. Use the completeness property of $\mathbb{R}$ to show that if $A \subseteq \mathbb{R}$ is non-empty and bounded below, then $\inf(A)$ exists.

   Hint: Consider the set $-A := \{-x : x \in A\}$. Show that $-A$ has a supremum and that $\inf(A) = \sup(-A)$.