Mathematical Analysis (MA552)

Practice Class Test II

Name:
Attempt ALL FOUR questions.

1. (a) Evaluate the following limit. Justify your answer.

$$\lim_{x\to 0} \frac{\log(\cos(x))}{x^2}.$$

(b) Hence evaluate

$$\lim_{x\to 0} \cos(x)^{x^{-2}}.$$

Justify your answer.

a) Let $f(x) = \log(\cos(x))$ and $g(x) = x^2$. These are both differentiable 2 on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and

$f(0) = \log(1) = 0 = g(0)$ 2, so by L'Hôpital 1

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \overset{2}{=} \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

providing the limit on RHS exists. 1

Now $\lim_{x\to 0} \frac{1}{\cos x} = 1$ by continuity of cos and

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ (famous limit!) 2

By the algebra of limits

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{-1}{2\cos x} \cdot \lim_{x\to 0} \frac{\sin x}{x} = -\frac{1}{2} \cdot 1 = -\frac{1}{2}$$

2

(Alternatively apply L'Hôpital twice.)

b) $\log\left(\cos(x)\right)^{x^{-2}} = \frac{\log(\cos(x))}{x^2} \longrightarrow -\frac{1}{2}$ as $x \to 0$

by first part. 3

By continuity of exp,

$\exp\left(\log\left(\cos(x)\right)^{x^{-2}}\right) = \cos(x)^{x^{-2}} \longrightarrow \exp\left(-\frac{1}{2}\right) = e^{-1/2}$

as $x \to 0$. 3

2. (a) State the Mean Value Theorem.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that $|f(b) - f(a)| \leq |b - a|$ for all $a, b \in \mathbb{R}$.

a) If $f : [a,b] \to \mathbb{R}$ is continuous 1 on $[a,b]$ and differentiable 1 on (a,b), then there exists $c \in (a,b)$ 1 such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad 2$$

b) The function f satisfies the conditions of the MVT on any interval $[a,b]$ 2

Hence, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}. \quad 1$$

As $|f'(c)| < 1$, we have $\left| \frac{f(b)-f(a)}{b-a} \right| < 1$ 2

and so $|f(b) - f(a)| < |b-a|$. 1

3. (a) Define the *radius of convergence* of a power series.

(b) Determine the radius of convergence for the following series:

$$\sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}.$$

(a). If a power series converges for all $|x| < R$ and diverges for all $|x| > R$, then R is the radius of convergence of the power series. 3

If the power series converges for all $x \in \mathbb{R}$, then its radius of convergence $R = \infty$. 1

If it diverges for all $x \neq 0$, then its radius of convergence $R = 0$ 1

(b) We use the Ratio Test. 1

Fix x and set $b_k = \frac{x^k}{\sqrt{k}}$.

Then $\left| \frac{b_{k+1}}{b_k} \right| = \left| \frac{x^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{x^k} \right| = \sqrt{\frac{k}{k+1}}\, |x|$

$\longrightarrow |x|$ as $k \to \infty$ 3

Hence $\sum_{k=1}^{\infty} b_k$ converges for $|x| < 1$ and diverges for $|x| > 1$. It's radius of convergence is $R = 1$. 2

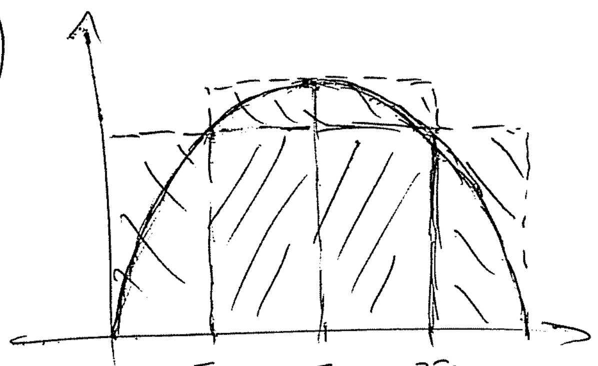
4. On $[0, \pi]$, let $f(x) = \sin(x)$ and let $\mathcal{P}$ be the partition

$$\mathcal{P}: \quad 0 < \frac{\pi}{4} < \frac{\pi}{2} < \frac{3\pi}{4} < \pi.$$

(a) Sketch the graph of the function f and indicate the rectangles whose areas contribute to the upper and lower Riemann sums.

(b) Determine the lower Riemann sum $L(f, \mathcal{P})$ and the upper Riemann sum $U(f, \mathcal{P})$.

a)



2

b)

$$L(f,P) = 0 \cdot \left(\frac{\pi}{4} - 0\right) + \frac{1}{\sqrt{2}} \cdot \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + 0 \cdot \left(\pi - \frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} = \frac{\pi}{2\sqrt{2}}$$

4

$$U(f,P) = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} - 0\right) + 1 \cdot \left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$+ 1 \cdot \left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + \frac{1}{\sqrt{2}}\left(\pi - \frac{3\pi}{4}\right)$$

$$= \left(2 + \frac{2}{\sqrt{2}}\right)\frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{2\sqrt{2}}$$

4