MA552 Analysis – Practice Class Test

NAME: November 10, 2006

Attempt all FOUR questions.

Q1. Prove from the definition of convergence that

$$\frac{3n^2+2}{9n^2+5} \to \frac{1}{3}$$

Answer Let $\epsilon > 0$ be given (1 mark), and setting

$$\underbrace{N(\epsilon) > \frac{1}{\sqrt{27\epsilon}}}\text{(1 mark: any correct N)}$$

we have for all $n > N$ (1 mark) that

$$\underbrace{\left|\frac{3n^2+2}{9n^2+5} - \frac{1}{3}\right| = \left|\frac{1}{3(9n^2+5)}\right|}_{\text{1 mark}} \underbrace{< \frac{1}{27n^2} < \frac{1}{27N^2} < \epsilon}_{\text{1 mark, any correct}}$$

as required.

Q2. Show the following limits. You may use any theorem proved in lectures or in the exercise sheets, but you must show that any hypotheses of the theorems used hold.

$$(i) \qquad \frac{4^n + 9^{n+1}}{4^{n+1} + 9^n} \to 9$$

$$(ii) \qquad \frac{4^n + n!}{5(n!) + 10^n} \to \frac{1}{5}$$

Answer

(i)

$$\lim \frac{4^n + 9^{n+1}}{4^{n+1} + 9^n} = \underbrace{\lim \frac{(4/9)^n + 9}{4(4/9)^n + 1} = \frac{\lim(4/9)^n + 9}{4\lim(4/9)^n + 1}}_{\text{(1 mark)}}$$

$$\to \frac{0 + 9}{0 + 1} = \frac{1}{9}$$

Using the Algebra of Limits Theorem, valid since all limits exist and the denominators are never zero, including in the limit (1 mark), and the fact that $c^n \to 0$ for $|c| < 1$(1 mark).

(ii)

$$\lim \frac{4^n + n!}{5(n!) + 10^n} \quad = \quad \frac{\lim(4^n)/n! + 1}{5 + \lim(10^n)/n!} \text{(1 mark)}$$

$$= \frac{0 + 1}{5 + 0} = \frac{1}{5}$$

using the Algebra of Limits Theorem, valid since all limits exist and the denominators are never zero, including in the limit, and the fact that $x^n/n! \to 0$ for all $x \in \mathbb{R}$ (1 mark)

Q3. Explain why the sequence

$$x_n = \frac{2 + \cos(n\pi)n}{2 + 3n}$$

does NOT converge.

Answer Note

$$x_n = \frac{(2/n) + (-1)^n}{(2/n) + 3}$$

As $n \to \infty$, $(2/n) \to 0$ and the sequence becomes close to the sequence $\frac{1}{3}(-1)^n$. (2 marks)

Then either

1 The picture is that x_n lies alternately close to near $\frac{1}{3}$ and $-\frac{1}{3}$. But if the sequence converged it would become close to only one number, namely the limit.

2 For $n = 2m$, the sequence x_{2m} converges to $1/3$, while for $n = 2m + 1$, the sequence x_{2m+1} converges to $-1/3$. But if the sequence converged, both these sequences would go to the same limit.

(3 marks)

Q4. Show that for $n > 4$,

$$0 < \frac{n^3}{5^n} < \frac{n^3}{4^4\binom{n}{4}} = \frac{6n^2}{4^3(n-1)(n-2)(n-3)}.$$

Hence show that

$$\frac{n^3}{5^n} \to 0.$$

Answer Let $n > 4$. Then since $5 = 1 + 4$, we have by the binomial theorem that

$$\underbrace{(1+4)^n = \sum_0^n \binom{n}{k} 4^k}_{\text{2 marks}} \underbrace{> \binom{n}{4} 4^4}_{\text{1 mark}}$$

as all terms are positive, and thus

$$0 < \frac{n^3}{5^n} < \frac{n^3}{4^4\binom{n}{4}} = \frac{n^3 4!}{4^4 n(n-1)(n-2)(n-3)}.$$

Then by the Sandwich Theorem (1 mark),

$$\frac{n^3}{5^n} \to 0$$

as it gets squeezed between zero and a sequence whose limit is zero.(1 mark, must indicate RHS goes to zero)

Notes Marks are awarded for the estimate, for knowing the two relevant theorems and putting it all together.