MA552 Analysis – Practice Class Test

NAME: No name, no marks November 5, 2009

Attempt all FIVE questions.

Q1. Prove from the definition of convergence that

$$\frac{4n + (-1)^n}{12n + 25} \to \frac{1}{3}$$

[5 marks]

Answer Let $\epsilon > 0$ be given (1 mark), and setting

$$N(\epsilon) > \frac{1}{\epsilon} \qquad \text{(1 mark: any correct N)}$$

we have for all $n > N$ ($\frac{1}{2}$ mark) that

$$\begin{aligned}
\left| \frac{4n + (-1)^n}{12n + 25} - \frac{1}{3} \right| &= \left| \frac{3(-1)^n - 25}{3(12n + 25)} \right| && (\tfrac{1}{2} \text{ mark}) \\
&< \frac{28}{3(12n)} && \text{by the Triangle Inequality (1 mark)} \\
&< \frac{1}{n} \\
&< \frac{1}{N} \\
&< \epsilon
\end{aligned}$$

as required. (1 mark, any correct sequence of inequalities)

Note: Use of Algebra of Limits Theorem instead: zero marks.

Q2. Find the limits of the following sequences. You may use any theorem proved in lectures or in the exercise sheets, but you must show that any hypotheses of the theorems used hold.

(i)

$$x_n = \frac{4^n + n!}{7^n + 5(n!)}$$

(ii)

$$x_n = \frac{(3n+1)^{15}}{\sqrt{2n}} \frac{\sqrt{7n+3}}{(5n+2)^{15}}$$

[5 marks]

Answer

(i)

$$\lim \frac{4^n + n!}{7^n + 5(n!)} = \frac{\lim \frac{4^n}{n!} + 1}{\lim \frac{7^n}{n!} + 5} \text{ (1 mark)}$$

$$= \frac{0+1}{0+5} = \frac{1}{5}$$

using the Algebra of Limits Theorem, valid since all limits exist and the denominators are never zero, including in the limit (1 mark), and the fact that $x^n/n! \to 0$ for all $x \in \mathbb{R}$ (1 mark)

(Space for Q2 continued)

Answer

(ii)

$$\begin{aligned}
x_n &= \frac{(3n+1)^{15}}{\sqrt{2n}} \frac{\sqrt{7n+3}}{(5n+2)^{15}} \\
&= \left(\frac{3n+1}{5n+2}\right)^{15} \sqrt{\frac{7n+3}{2n}} \\
&= \left(\frac{3+\frac{1}{n}}{5+\frac{2}{n}}\right)^{15} \sqrt{\frac{7+\frac{3}{n}}{2}} \text{ (1 mark)} \\
&\to \left(\frac{3}{5}\right)^{15} \sqrt{\frac{7}{2}}
\end{aligned}$$

using the Algebra of Limits Theorem, valid since all limits exist and the denominators are never zero, including in the limit, and the fact that if $y_n > 0$ and $y_n \to \ell$ then $\sqrt{y_n} \to \sqrt{\ell}$ (1 mark).

Q3. Consider the sequence

$$x_n = \sqrt{2n^2+5} - \sqrt{2n^2-(-1)^n n}.$$

By rationalising the numerator or otherwise, decide whether the sequence $\{x_n\}$ converges, or not. If it converges, give a proof. If it does not converge, say why it does not.

[5 marks]

Answer Rationalising the numerator yields

$$\begin{aligned} x_n &= \frac{5+(-1)^n n}{\sqrt{2n^2+5}+\sqrt{2n^2-(-1)^n n}} \\ &= \frac{\frac{5}{n}+(-1)^n}{\sqrt{2+\frac{5}{n^2}}+\sqrt{2-(-1)^n\frac{1}{n}}} \end{aligned}$$

(2 marks)

The sequence does NOT converge.

Possible justifications:

1 The sequence becomes close to the sequence $(-1)^n/2\sqrt{2}$ which does not converge.

2 The picture is that x_n lies alternately close to near $1/2\sqrt{2}$ or $-1/2\sqrt{2}$. But if the sequence converged it would become close to only one number, namely the limit.

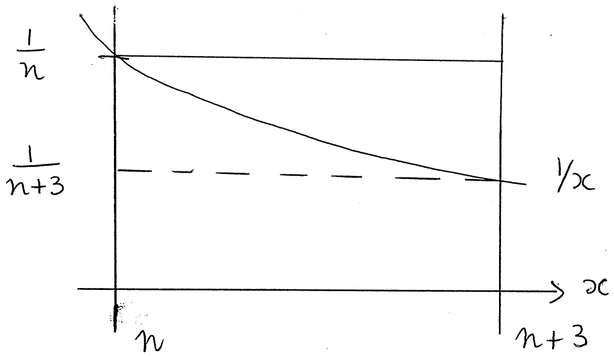
3 For $n = 2m$, the sequence x_{2m} converges to $1/2\sqrt{2}$, while for $n = 2m+1$, the sequence x_{2m+1} converges to $-1/2\sqrt{2}$. But if the sequence converged, both these subsequences would go to the same limit.

(3 marks for a sentence giving the idea that there needs to be one unique limit for a convergent sequence, so [1] attracts only 1 mark.)

Q4. Draw the graph of $1/x$ on the interval $[n, n+3]$. By comparing the area under the curve with both the upper and lower areas, show that

$$n \log\left(\frac{n+3}{n}\right) \to 3,$$

justifying your reasoning. [5 marks]



[2 marks correct, fully labelled diagram]

$$\text{lower area} < \int_n^{n+3} \frac{\mathrm{d}x}{x} < \text{upper area}$$

or

$$\frac{3}{n+3} < \log(n+3) - \log n < \frac{3}{n}$$

so that

$$\underbrace{\frac{3n}{n+3}}_{} < n \log\left(\frac{n+3}{n}\right) < 3 \qquad (1)$$

$$\to 3 \qquad (2)$$

(2 marks) By the Sandwich Theorem (1 mark),

$$n \log\left(\frac{n+3}{n}\right) \to 3.$$

Q5. (a) What does ϵ denote in Real Analysis?

1. An arbitrary positive number.
2. An arbitrarily small positive number.
3. $1/10^N$ or $1/2^N$ for some N.
4. I don't know, no one will tell me.

(b) What is the main purpose of studying proofs in Real Analysis?

1. To waste time and bore students.
2. To give the lecturer a chance to show off.
3. To convince students the result is true.
4. To induct students in to what it means to be a mathematician.

1 bonus mark each