Mathematical Analysis (MA552)

Class Test II

Name:

Attempt ALL FOUR questions.
Allowable calculators: Casio fx-83 and Casio fx-85

1. Evaluate the following limits. Justify your answers.

$$\text{(a)} \lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} \qquad \text{(b)} \lim_{x\to 0} \frac{3^{2x}-1}{\sin(x)}.$$

Hint for (b): Recall that $a^x = e^{x \log a}$. [each 8 marks]

a) Let $f(x) = x - \sqrt{x}$ and $g(x) = x - 1$.
Both are differentiable 2 for $x > 0$ and 1
$f(1) = g(1) = 0$ 2 By L'Hôpital's rule 1 and continuity of $\sqrt{x}$

$$\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} = \lim_{x\to 1} \frac{1-\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2} \quad 2$$

Alternatively: $\dfrac{x-\sqrt{x}}{x-1} = \dfrac{\sqrt{x}(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)} = \dfrac{\sqrt{x}}{\sqrt{x}+1}$ 5

Hence, $\displaystyle\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}}{\sqrt{x}+1} = \frac{1}{2}$ 2

by continuity of $\sqrt{x}$ at 1. 1

b) Let $f(x) = 3^{2x} - 1$ and $g(x) = \sin x$.

Both are differentiable ② and $f(0) = g(0) = 0$ ②

By L'Hôpital's Rule ①

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

Now
$$f'(x) = \frac{d}{dx}\left(e^{2x\log 3} - 1\right) = 2\log 3\, e^{2x\log 3}$$
$$= 2\log 3 \cdot 3^{2x} \quad ①$$

So
$$\lim_{x\to 0} \frac{3^{2x}-1}{\sin x} = \lim_{x\to 0} \frac{2\log 3 \cdot 3^{2x}}{\cos x} = 2\log 3 \quad ①$$

by continuity of the exponential function and cos. ①

2. (a) State Taylor's Theorem. [7 marks]

(b) Let $f(x) = \sqrt{1+x}$. Find the Taylor polynomial of degree 3 for f at $x = 1$. [[illegible] marks]

a) Suppose that $f, f', \dots, f^{(n)}$ are 1 continuous on $[a,x]$ and $f^{(n+1)}$ exists 1 on (a,x). Then there exists $c \in (a,x)$ 1 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad 2 \quad + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad 2 .$$

b) We have $f'(x) = \frac{1}{2\sqrt{1+x}}$ 1, $f''(x) = \frac{-1}{4(1+x)^{3/2}}$ 1

$$f'''(x) = \frac{3}{8(1+x)^{5/2}} \quad 1$$

Hence,

$$p_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) + \frac{-1}{8\cdot\sqrt{2}\cdot 2!}(x-1)^2$$

$$+ \frac{3}{8\cdot 2^{5/2}\cdot 3!}(x-1)^3$$

$$= \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) - \frac{1}{16\sqrt{2}}(x-1)^2 + \frac{1}{64\sqrt{2}}(x-1)^3$$

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3. (a) Find all functions satisfying $f'(x) = x^5$. [5 marks]

(b) Show that if $f'(x)f(x) - 1 = 0$, then $f(x) = \pm\sqrt{2x+c}$ for some $c \in \mathbb{R}$. [7 marks]

Justify your answers.

a) Consider $g(x) = f(x) - \frac{x^6}{6}$. 2

Then $g'(x) = f'(x) - x^5 = 0$ 1

Hence, $g(x) = c$ is constant 1 and

$f(x) = g(x) + \frac{x^6}{6} = c + \frac{x^6}{6}$, $c \in \mathbb{R}$. 1

b) We solve the equation of f for c:

$f^2(x) - 2x = c$.

Define $g(x) = f^2(x) - 2x$ 4. We want to show that g is constant.

Now, $g'(x) = 2f(x)f'(x) - 2 = 0$ 1, so

g is constant 1 and $f^2(x) - 2x = +c$, or

$f(x) = \pm\sqrt{2x+c}$. 1

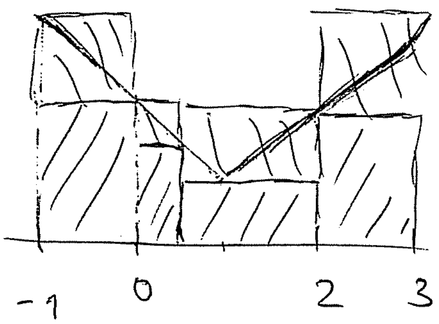
4. On $[-1,3]$, let $f(x) = |x-1| + 1$ and let $\mathcal{P}$ be the partition

$$\mathcal{P}: \quad -1 < 0 < \frac{1}{2} < 2 < 3.$$

(a) Sketch the graph of the function f and indicate the rectangles whose areas contribute to the upper and lower Riemann sums. [2 marks]

(b) Determine the lower Riemann sum $L(f,\mathcal{P})$ and the upper Riemann sum $U(f,\mathcal{P})$. [~~8~~ 6 marks]

a)



2

b) $$L(f,P) = 2\cdot(0-(-1)) + \frac{3}{2}\left(\frac{1}{2}-0\right) + 1\left(2-\frac{1}{2}\right) + 2(3-2)$$

$$= 2 + 3/4 + 3/2 + 2 = \frac{25}{4}$$ 3

$$U(f,P) = 3\cdot(0-(-1)) + 2\cdot\left(\frac{1}{2}-0\right) + 2\left(2-\frac{1}{2}\right) + 3(3-2)$$

$$= 3 + 1 + 3 + 3 = 10$$ 3