Mathematical Analysis Worksheet 9

Bounds, Suprema and Infima

Let $A \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is an *upper bound* for A if $M \geq a$ for all $a \in A$. $m \in \mathbb{R}$ is a *lower bound* for A if $m \leq a$ for all $a \in A$.
If $a \in A$ is an upper bound for A, then a is the *maximum* of A. If $a \in A$ is a lower bound for A, then a is the *minimum* of A. Note that a maximum or minimum has to lie in the set itself. Not every bounded set has a maximum or a minimum, e.g. $A = (1, 2)$.

Example 1. *Question: Show that $(0, \infty)$ is unbounded above (i.e. it has no upper bound).*
Answer: We prove this by contradiction. Assume $a \in \mathbb{R}$ is an upper bound for $(0, \infty)$. Then $a > 0$ and $a + 1$ is a positive number (hence it lies in $(0, \infty)$), which is strictly larger than a, so a cannot be an upper bound for $(0, \infty)$.

Example 2. *Question: Show that $(1, 2)$ has no maximum.*
Answer: We prove this by contradiction. Assume $a = \max(1, 2)$. As $a \in (1, 2)$, we have $1 < a < 2$. Consider $b := a + (2 - a)/2$. Then $b > a > 1$ and

$$b - 2 = a + \frac{2 - a}{2} - 2 = \frac{a}{2} - 1 < 0.$$

Therefore $b < 2$, so $b \in (1, 2)$ and $b > a$ which contradicts $a = \max(1, 2)$. Hence, $(1, 2)$ has no maximum.

Proofs of existence of a maximum are often very easy:

Example 3. *Question: Show that $(1, 2]$ has a maximum.*
Answer: We claim that $\max(1, 2] = 2$. Clearly, $2 \in (1, 2]$ and if $a \in (1, 2]$, then $a \leq 2$, so 2 is an upper bound and hence the maximum.

On the other hand, the **Extreme Values Theorem** is a fairly deep result:
Let f be continuous on a closed bounded interval $[a, b]$. Then f attains its bounds.

Example 4. *Question: True or false; prove the following statement or give a counterexample.*

1. *Let $f : (a, b) \to \mathbb{R}$ be continuous. There exists $c \in (a, b)$ such that $f(c) \geq f(x)$ for all $x \in (a, b)$.*
2. *Let $f : [a, b] \to \mathbb{R}$ be continuous. There exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.*

Answer:

1. *This is false, e.g. $f(x) = x$ on $(1, 2)$ is a counterexample. Clearly, f is continuous, but as seen above, $(1, 2)$ has no maximum, so f does not attain its upper bound.*
2. *This is true, as f attains its maximum according to the Extreme Values Theorem.*

To get around the difficulty of having bounded sets without a maximum or a minimum, we introduce two new notions: the supremum and the infimum.
Let $A \subseteq \mathbb{R}$. Then $M \in \mathbb{R}$ is the *supremum* or *least upper bound* of A, if

1. M is an upper bound for A and
2. for any other upper bound M_0 of A, we have $M \leq M_0$.

A number $m \in \mathbb{R}$ is the *infimum* or *greatest lower bound* of A, if

1. m is a lower bound for A and
2. for any other lower bound m_0 of A, we have $m \geq m_0$.

Example 5. *Question: Classify the following subsets of $\mathbb{R}$ as bounded or unbounded. If the set is bounded, write down the supremum and infimum.*

$$\text{(i) } \{x : |x-5| \leq 2\}; \quad \text{(ii) } \{1/n^2 : n \in \mathbb{N}\};$$
$$\text{(iii) } \{1/x^2 : x \in \mathbb{R} \setminus \{0\}\}; \quad \text{(iv) } \bigcup_{n \text{ prime}} (-1/n, 1/n).$$

(There is no need to provide proofs of your assertions.)
Answer: (i) As the set is the closed interval $[3,7]$, it is bounded with $\inf[3,7] = 3$ and $\sup[3,7] = 7$.
(ii) The set is bounded with infimum 0 and supremum 1.
(iii) The set is unbounded.
(iv) The set is bounded with infimum $-1/2$ and supremum $1/2$

Example 6. *Question: Prove that $\sup(1,2) = 2$.*
Answer: Clearly $2 \geq x$ for any $x \in (1,2)$, so 2 is an upper bound. We prove by contradiction that it is the least upper bound. Assume $a < 2$ is a smaller upper bound. Then as $a > 3/2$, we have in particular that $a > 1$, so $a \in (1,2)$. Consider $b := a + (2-a)/2$. Then $b > a > 1$ and

$$b - 2 = a + \frac{2-a}{2} - 2 = \frac{a}{2} - 1 < 0.$$

Therefore $b < 2$, so $b \in (1,2)$ which contradicts that a is an upper bound. Hence, $\sup(1,2) = 2$.

Exercises 7. *1. (a) Show that $(-\infty, 0)$ is unbounded below.*
(b) Show that $(-\infty, 0)$ has no maximum.
(c) Show that $\mathbb{N}$ is unbounded above.
(d) Show that $\mathbb{N}$ has a minimum.

2. True or false; prove the following statement or give a counterexample.

(a) Let $f : (a,b) \to \mathbb{R}$ be bounded. There exists $c \in (a,b)$ such that $f(c) \geq f(x)$ for all $x \in (a,b)$.
(b) Let $f : [a,b] \to \mathbb{R}$ be bounded. There exists $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$.
(c) Let $f : (a,\infty) \to \mathbb{R}$ be continuous. There exists $c \in (a,\infty)$ such that $f(c) \geq f(x)$ for all $x \in (a,\infty)$.
(d) Let $f : [a,\infty) \to \mathbb{R}$ be contnuous and bounded. There exists $c \in [a,\infty)$ such that $f(c) \geq f(x)$ for all $x \in [a,\infty)$.

3. Classify the following subsets of $\mathbb{R}$ as bounded or unbounded. If the set is bounded, write down the supremum and infimum.

$$\text{(i) } \{x : |x+2| < 2\}; \quad \text{(ii) } \{n^2 : n \text{ is prime}\};$$
$$\text{(iii) } \{1/x^2 : x \in \mathbb{R} \setminus [-1,1]\}; \quad \text{(iv) } \bigcap_{n=1}^{\infty} (1 - 1/n, 3 + 1/n).$$

(There is no need to provide proofs of your assertions.)

4. Prove that $\inf(1,2) = 1$ and that $\inf[1,2) = 1$.