Mathematical Analysis Worksheet 8

IVT, MVT and all that jazz

Recall the
Intermediate Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous and $k \in (f(a), f(b))$ or $k \in (f(b), f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = k$.
Notes:

1. This result is very useful to prove existence of solutions to equations.
2. It does not tell us where a solution lies in the interval and there may be more than one solution.

Example 1. *Question: Show that* $x^2 = \cos(x)$ *has a solution in* $[0, \pi/2]$.
Answer: We consider $f(x) = x^2 - \cos(x)$. *Then* f *is continuous and* $f(0) = -1$ *and* $f(\pi/2) = \pi^2/4$. *By the IVT, there exists* $c \in (0, \pi/2)$ *such that* $f(c) = 0$, *i.e.* $c^2 = \cos(c)$.

If a problem is in the form $x = f(x)$ or can easily be rewritten in this way, it is worth checking the **Fixed Point Theorem**: Let $f : [a, b] \to [a, b]$ be continuous. Then f has a fixed point.

Example 2. *Question: Show that* $x = \cos(x)$ *has a solution in* $[0, 1]$.
Answer: We consider $f(x) = \cos(x)$ *on* $[0, 1]$. *Then* f *is continuous and as* $0 \leq \cos(x) \leq 1$ *for all* $x \in [0, 1]$, f *has a fixed point in* $[0, 1]$.

To prove uniqueness we need an additional argument which either relies on monotonicity or
Rolle's Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable on (a, b) with $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Example 3. *Question: Show that* $x^2 = \cos(x)$ *has a unique solution in* $[0, \pi/2]$.
Answer: We first show existence as above, introducing the continuous function f *and using the IVT.*
To prove uniqueness, we argue by contradiction. Assume we have two solutions $x_1 < x_2 \in (0, \pi/2)$, *i.e.* $f(x_1) = f(x_2) = 0$. *Since* f *is differentiable, we can apply Rolle's Theorem on the interval* (x_1, x_2) *which guarantees existence of* $c \in (x_1, x_2)$ *such that* $f'(c) = 0$. *However,*

$$f'(x) = \underbrace{2x}_{>0} + \underbrace{\sin(x)}_{>0} > 0 \qquad \textit{for } x \in (0, \pi/2),$$

giving a contradiction. Hence there is only one solution to $f(x) = 0$.
Alternatively, we could prove uniqueness by saying that f *is differentiable on* $(0, \pi/2)$ *and that* $f'(x) > 0$ *on the interval. Therefore,* f *is monotonically increasing and can only have one zero.*

Rolle's Theorem can be generalised to the
Mean Value Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
As well as proving uniqueness results, the MVT can also be used to get estimates on functions.

Example 4. *Question: Apply the MVT to* $f(x) = e^{-x}$ *to prove that* $e^{-x} > 1 - x$ *for* $x > 0$.
Answer: Let $x > 0$. *The function* $f(x) = e^{-x}$ *is differentiable, so we can apply the MVT to* f *on the interval* $(0, x)$. *We have* $f'(x) = -e^{-x}$, *so there exists* $c \in (0, x)$ *such that* $-e^{-c} = \frac{e^{-x}-1}{x}$. *Thus* $xe^{-c} = 1 - e^{-x}$. *As* $0 < e^{-c} < 1$, *we get the estimate* $x > 1 - e^{-x}$ *which proves that* $e^{-x} > 1 - x$.

Note: We could have also used the more general result (cf. Exercise Sheet 4) that if $|f'(x)| < M$ for all x, then $|f(b) - f(a)| < M(b - a)$.
Another consequence of the MVT is that a function is constant if and only if its derivative vanishes at all points. This allows us to find *all* solutions to simple differential equations.

Example 5. *Question: Find all solutions to* $f'(x) = 2\cos(x)$. *Justify your answer.*
Answer: Consider $g(x) = 2\sin(x) - f(x)$. *Then* g *is differentiable and* $g'(x) = 2\cos(x) - 2\cos(x) = 0$, *so* g *must be equal to a constant,* c *say. Hence,* $c = 2\sin(x) - f(x)$ *or* $f(x) = 2\sin(x) - c$ *for any* $c \in \mathbb{R}$.

Notes:

1. To find the function g, you need to "guess" it or do a "separation of variables calculation" as you saw in courses on differential equations.

2. As this is an Analysis rather than a Calculus course, to justify your solution, you do need to find a function g which is differentiable and show that its derrivative is zero.

Exercises 6. *1. (a) Show that* $x^2 = 2 + \log(x)$ *has a unique solution in* $(1, 2)$.
 (b) Show that $x^3 - 4x^2 + \cos(x) = 0$ *has a unique solution in* $(0, 1)$.
 (c) Show that $x = 2 + \log(x)$ *has a unique solution in* $(2, 4)$.

2. (a) Consider the function $f(x) = \sqrt{x}$ *for* $x > 0$. *Show that* $f'(x)$ *is a decreasing function on* $(0, \infty)$. *Apply the Mean Value Theorem to* f *on the interval* $[100, 102]$ *to prove*

$$10 + \frac{1}{11} < \sqrt{102} < 10 + \frac{1}{10}.$$

 (b) Apply the MVT to $f(x) = \arcsin(x)$ *on the interval* $[0.5, 0.6]$ *to show that*

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \arcsin(0.6) < \frac{\pi}{6} + \frac{1}{8}.$$

 (Use that $\pi/6 = \arcsin(0.5)$.*)*

3. (a) Find all functions f *satisfying* $f'(x) = 1/\sqrt{1 - x^2}$ *for* $|x| < 1$.
 (b) Find all functions f *satisfying* $f'(x) = \sin(x) + \sin(2x)$.
 (c) Prove that if $f'(x) = af(x)$ *for all* $x \in \mathbb{R}$ *and some constant* $a \in \mathbb{R}$, *then* $f(x) = Ce^{ax}$ *for some* $C \in \mathbb{R}$.