Mathematical Analysis Worksheet 7

Differentiation - Definition

Recall that a function $f : [a, b] \to \mathbb{R}$ is differentiable at $c \in (a, b)$ if

$$f'(c) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists.}$$

Notes:

1. We only define the derivative at points c inside the interval (a, b) so that we can approach c from both sides (i.e. h can be both positive and negative).
2. A function is differentiable on the interval (a, b) if it is differentiable at all points c of the interval. In this case, it can be important to distinguish between $f'(c)$, the value of the derivative at point c, and the function f', the derivative of f.

Example 1. *Question: Show that $f(x) = \sqrt{x}$ is differentiable at $x = 1$.*
Answer: We consider

$$\begin{aligned}\frac{f(1+h) - f(1)}{h} &= \frac{\sqrt{1+h} - \sqrt{1}}{h} = \frac{(\sqrt{1+h} - \sqrt{1})(\sqrt{1+h} + \sqrt{1})}{h(\sqrt{1+h} + \sqrt{1})} \\ &= \frac{1+h-1}{h(\sqrt{1+h} + \sqrt{1})} = \frac{1}{\sqrt{1+h} + 1}.\end{aligned}$$

As $g(y) = \frac{1}{1+\sqrt{1+y}}$ is continuous at $y = 0$, we get

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{1}{1 + \sqrt{1+h}} = \frac{1}{1+\sqrt{1}} = \frac{1}{2}.$$

Hence the limit exists and f is differentiable at $x = 1$.

To show that a function is differentiable, we argue in the same way:

Example 2. *Question: Show that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$.*
Answer: Let $x > 0$. We consider

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.\end{aligned}$$

As the functions $g_x(y) = \frac{1}{\sqrt{x+y}+\sqrt{x}}$ are continuous at $y = 0$ for every $x > 0$, we get

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Hence the limit exists for all $x > 0$ and f is differentiable with derivative $f'(x) = \frac{1}{2\sqrt{x}}$.

Often functions are defined piecewise and you may be asked to check for continuity or differentiablity where two pieces meet. This will require you to use the definition of the derivative. To calculate the derivative at other points, unless specifically required otherwise, you can use the usual rules of calculus (state which ones you're using!)

Example 3. *Question (from 2009 exam): Consider* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

Show that f *is differentiable at* 0*. Calculate* $f'(x)$ *for* $x \neq 0$ *using the usual rules of calculus. Is* f' *continuous at* $x = 0$*?*
Answer: To calculate the derivative at 0*, we consider*

$$\frac{f(h) - f(0)}{h} = \frac{h^2 \cos\left(\frac{1}{h}\right) - 0}{h} = h \cos\left(\frac{1}{h}\right).$$

Since $\left|\cos\left(\frac{1}{h}\right)\right| \leq 1$*, we have*

$$\left|h \cos\left(\frac{1}{h}\right)\right| \leq |h| \to 0 \text{ as } h \to 0.$$

Hence, $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = 0$ *and* f *is differentiable at* 0 *with* $f'(0) = 0$*.*
For $x \neq 0$*, by the product and the chain rule, we get*

$$f'(x) = 2x \cos\left(\frac{1}{x}\right) + x^2 \left(-\frac{1}{x^2}\right)\left(-\sin\left(\frac{1}{x}\right)\right) = 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

Using the limit characterisation of continuity, we see that $g(x) = 2x \cos\left(\frac{1}{x}\right)$ *is continuous at* 0*, since by the same argument as above,* $|g(x)| \leq 2|x| \to 0$ *as* $x \to 0$*. However,* $h(x) = \sin\left(\frac{1}{x}\right)$ *is not continuous at* 0*. To see this, consider the sequence* $x_n = \frac{1}{(n+1/2)\pi}$*. Then* $x_n \to 0$ *as* $n \to \infty$*, but* $h(x_n) = \sin(n + 1/2)\pi$ *alternates between* ± 1*, so does not converge. Therefore,* h *is not continuous at* 0 *and so neither is* f' *(else* $h = f' - g$ *would have to be continuous as the sum of continuous functions).*

Exercises 4.

1. *Using the definition of differentiablility, show that the following functions are differentiable at* $x = 1$ *and determine their derivatives.*
   (a) $f(x) = x^2 - 5x + 1$,
   (b) $g(x) = \sqrt{x+5}$,
   (c) $h(x) = \frac{1}{x^2+1}$.

   Show that all three functions are differentiable as functions defined on $(0, \infty)$*.*

2. *Check whether the following functions are differentiable at* 0*. If so, calculate* $f'(0)$ *and* $f'(x)$ *for* $x \neq 0$ *and determine whether* f *is continuous at* 0*.*

   (a)
   $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (b)
   $$f(x) = \begin{cases} \frac{1}{x} \sin(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (c) *For some twice differentiable function* g *satisfying* $g(0) = g'(0) = g''(0) = 0$*, consider*
   $$f(x) = \begin{cases} g(x) \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (Hint: Use L'Hôpital's Rule.)