Mathematical Analysis Worksheet 6

Continuity - Application of Theorems and L'Hôpital's Rule

To prove continuity of functions, there are two particularly useful theorems: One states that sums, products etc. of continuous functions are continuous. The other is the limit characterisation of continuity: $f : [a, b] \to \mathbb{R}$ is continuous at $x_0 \in [a, b]$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in $[a, b]$ with $x_n \to x_0$ as $n \to \infty$ we have $f(x_n) \to f(x_0)$ as $n \to \infty$.

Example 1. *Question: Using only that* 1 *and* x *are continuous, prove that* $f(x) = \frac{x^3+2x-1}{x^2+1}$ *is continuous as a function from* $\mathbb{R}$ *to* $\mathbb{R}$.
Answer: Since multiples of continuous functions are continuous, -1 *and* $2x$ *are continuous.*
As the product of continuous functions, x^2 *is continuous, as is* x^3.
Sums of continuous functions are continuous, so $x^3 + 2x - 1$ *and* $x^2 + 1$ *are continuous.*
Quotients of continuous functions are continuous if the denominator is not zero. As $x^2 + 1 \neq 0$ *for all* $x \in \mathbb{R}$*, the function* f *is continuous.*

L'Hôpital's Rule may sometimes come in useful in questions involving the limit characterisation of continuity - be careful to check that you may apply it!

Example 2. *Question: Determine the constant* K *so that the following function defined on* $(-1, 1)$ *is continuous at* $x = 0$.

$$f(x) = \begin{cases} \frac{\arcsin(x)-x}{x^3} & \textit{if } x \neq 0, \\ K & \textit{if } x = 0. \end{cases}$$

Answer: We use the limit characterisation of continuity, i.e. f *is continuous at* 0 *if we choose*

$$K = \lim_{x\to 0} f(x).$$

Let $g(x) = \arcsin(x) - x$ *and* $h(x) = x^3$*. Then both are differentiable and* $g(0) = h(0) = 0$*, so we can apply L'Hôpital's Rule to get,*

$$\lim_{x\to 0} \frac{\arcsin(x) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2}.$$

Both the numerator and the denominator are differentiable and vanish at $x = 0$*, so we again apply L'Hôpital's Rule to get*

$$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} = \lim_{x\to 0} \frac{\frac{x}{(1-x^2)^{3/2}}}{6x} = \lim_{x\to 0} \frac{1}{6(1-x^2)^{3/2}} = \frac{1}{6}.$$

Here, the last equality follows from the fact that $k(x) = \frac{1}{6(1-x^2)^{3/2}}$ *is continuous at* $x = 0$ *and the limit characterisation of continuity. Hence,* f *is continuous if (and only if) we choose* $K = 1/6$.

The limit characterisation of continuity is often useful for proving that a function is not continuous:

Example 3. *Question: Let $g : \mathbb{R} \to \mathbb{R}$ be defined by*

$$g(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 0, \\ \sin(x) & \text{if } x > 0. \end{cases}$$

Prove that g is discontinuous at 0.
Answer: Clearly, $g(0) = -3$. Moreover,

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \sin(x) = 0.$$

(This follows from $|\sin(x)| \leq |x|$.) Hence $g(0) \neq \lim_{x \to 0^+} g(x)$ and g is not continuous at 0 by the limit characterisation of continuity.
Notes:

1. *In fact, in this example $\lim_{x \to 0} g(x)$ does not exist as the left and right limit are different.*
2. *To make things more explicit, rather than looking at the right limit, it's also possible to choose a sequence, e.g. $x_n = 1/n$ and argue that $x_n \to 0$, but $g(x_n) \not\to g(0)$.*

Exercises 4.

1. *Using only that 1, x and $\sin(x)$ are continuous on $\mathbb{R}$, show that the following functions are continuous on $(-\pi, \pi)$:*

$$f(x) = \frac{3 - \sin^2(x) + 3x^6}{2 + \sin(x)}, \qquad g(x) = \cos(x), \qquad h(x) = \frac{2\sin^3(x) - 1}{1 + \tan^2(x)}.$$

2. *For which value of the constant K are the following functions continuous at 0?*

$$f(x) = \begin{cases} \frac{\sin(x) - x}{x^3} & \text{if } x \neq 0, \\ K & \text{if } x = 0, \end{cases}$$

$$g(x) = \begin{cases} \frac{\log(1+x) - x}{x^2} & \text{if } x \neq 0, \\ K & \text{if } x = 0, \end{cases}$$

$$h(x) = \begin{cases} (\frac{1}{2}(a^x + 1))^{1/x} & \text{if } x \neq 0, \\ K & \text{if } x = 0. \end{cases}$$

3. *Prove that the following functions are discontinuous at $x = 1$.*

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x < 1, \\ x - 1 & \text{if } x \geq 1, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$