Mathematical Analysis Worksheet 5

The $(\varepsilon, \delta)$-definition of continuity

We recall the definition of continuity: Let $f : [a, b] \rightarrow \mathbb{R}$ and $x_0 \in [a, b]$. $f$ is continuous at $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

We sometimes indicate that the $\delta$ may depend on $\varepsilon$ by writing $\delta(\varepsilon)$.

As with convergence of sequences, all proofs of continuity of functions using the definition follow a fixed format.

**Example 1.** Question: Prove that $f(x) = x^3 + 2x - 1$ is continuous at $x = 1$.

**Answer:** Let $\varepsilon > 0$ be given. Let $0 < \delta < \min\{1, \varepsilon/9\}$. Then for $|x - 1| < \delta$ we have

$$|f(x) - f(1)| = |x^3 + 2x - 1 - 2| = |x^3 + 2x - 3| = |x - 1||x^2 + x + 3|.$$  

Since $\delta < 1$, and $|x - 1| < \delta$ we have $x \in (0, 2)$, so $|x^2 + x + 3| < 9$. Therefore,

$$|f(x) - f(1)| < 9\delta < \varepsilon.$$  

**Notes:**

1. The wording in bold always appears in the proof of continuity using the $(\varepsilon, \delta)$-definition.

2. The sequence of equalities and inequalities ending in “$< \varepsilon$” depends on the function being studied. This is where your skill and experience come in. The aim is to always to extract a positive power of $\delta$.

3. The value of $\delta$ is filled in last, when you know what it has to be from your sequence of inequalities ending in “$< \varepsilon$”.

4. There are many ways of choosing $\delta$. In this proof, we estimated $|x^2 + x + 3|$ for $x$ near 1. This required putting an upper bound on $\delta$ (independent of $\varepsilon$). Here, we chose $\delta < 1$.

   An obvious alternative is to write

   $$|x^3 + 2x - 1 - 2| = |x - 1| \left|(x - 1)^2 + 3(x - 1) + 5\right| < \delta^3 + 3\delta^2 + 5\delta.$$  

   Then we choose $\delta$ so that all three terms on the right hand side are less than $\varepsilon/3$, i.e. $\delta < \min\left\{\frac{\sqrt{\varepsilon}}{3}, \frac{\sqrt{\varepsilon}}{3}, \frac{\varepsilon}{15}\right\}$.

**Exercises 2.** Using the $(\varepsilon, \delta)$-definition of continuity, show that the following functions are continuous at $x = 2$.

1. $f(x) = x^2 - 5x + 1$,
2. $g(x) = \sqrt{x}$,
3. $h(x) = \frac{1}{x^2 + 1}$.
Discontinuous functions

To show from the \((\varepsilon, \delta)\)-definition of continuity that a function is discontinuous at a point \(x_0\), we need to negate the statement: “For every \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(|x - x_0| < \delta\) implies \(|f(x) - f(x_0)| < \varepsilon\).”

Its negative is the following (check that you understand this!):
“There exists an \(\varepsilon > 0\) such that for every \(\delta > 0\) there exists a point \(x\) with \(|x - x_0| < \delta\) and \(|f(x) - f(x_0)| \geq \varepsilon\).”

Example 3. Question: Let \(g : \mathbb{R} \to \mathbb{R}\) be defined by

\[
g(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 0, \\ \sin(x) & \text{if } x > 0. \end{cases}
\]

Prove that \(g\) is discontinuous at \(0\).

Answer: Let \(\varepsilon = 2\) and let \(\delta > 0\) be given. Choose \(x \in (0, \delta)\) with \(x < \pi\). Then \(g(x) = \sin(x) > 0\). Hence,

\[|x - 0| < \delta \quad \text{and} \quad |g(x) - g(0)| = |\sin(x) - 3| > 3 \geq \varepsilon\]

and \(g\) is discontinuous at \(0\).

Notes:

1. The wording in bold always appears in the proof of discontinuity using the \((\varepsilon, \delta)\)-definition.
2. It’s probably good to sketch the graph of the function.
3. Think about how big the gap is. This determines which \(\varepsilon\) we can choose, here we could have chosen anything up to 3.
4. You will need to choose \(x\), so that it’s in a suitable range (here we chose \(0 < x < \pi\)).

Exercises 4. Prove that the following functions are discontinuous at \(x = 1\).

\[
f(x) = \begin{cases} x^2 - 3 & \text{if } x < 1, \\ x - 1 & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1/(x - 1) & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}
\]