Mathematical Analysis Worksheet 5

The (ε, δ)-definition of continuity

We recall the definition of continuity: Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. f is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.
We sometimes indicate that the δ may depend on ε by writing $\delta(\varepsilon)$.
As with convergence of sequences, all proofs of continuity of functions using the definition follow a fixed format.

Example 1. *Question: Prove that* $f(x) = x^3 + 2x - 1$ *is continuous at* $x = 1$.
Answer: ***Let*** $\varepsilon > 0$ ***be given. Let*** $0 < \delta < \min\{1, \varepsilon/9\}$***. Then for*** $|x - 1| < \delta$ ***we have***

$$\begin{aligned} |f(x) - f(1)| &= |x^3 + 2x - 1 - 2| = |x^3 + 2x - 3| \\ &= |x - 1||x^2 + x + 3|. \end{aligned}$$

Since $\delta < 1$, *and* $|x - 1| < \delta$ *we have* $x \in (0, 2)$, *so* $|x^2 + x + 3| < 9$. *Therefore,*

$$|f(x) - f(1)| < 9\delta < \varepsilon.$$

Notes:

1. *The wording in bold always appears in the proof of continuity using the* (ε, δ)*-definition.*
2. *The sequence of equalities and inequalities ending in "*$< \varepsilon$*" depends on the function being studied. This is where your skill and experience come in. The aim is to always to extract a positive power of* δ.
3. *The value of* δ *is filled in last, when you know what it has to be from your sequence of inequalities ending in "*$< \varepsilon$*".*
4. *There are many ways of choosing* δ. *In this proof, we estimated* $|x^2 + x + 3|$ *for* x *near* 1. *This required putting an upper bound on* δ *(independent of* ε*). Here, we chose* $\delta < 1$.
   An obvious alternative is to write

   $$|x^3 + 2x - 1 - 2| = |x - 1| \left|(x - 1)^2 + 3(x - 1) + 5\right| < \delta^3 + 3\delta^2 + 5\delta.$$

   Then we choose δ *so that all three terms on the right hand side are less than* $\varepsilon/3$, *i.e.* $\delta < \min\left\{\sqrt[3]{\frac{\varepsilon}{3}}, \frac{\sqrt{\varepsilon}}{3}, \frac{\varepsilon}{15}\right\}$.

Exercises 2. *Using the* (ε, δ)*-definition of continuity, show that the following functions are continuous at* $x = 2$.

1. $f(x) = x^2 - 5x + 1$,
2. $g(x) = \sqrt{x}$,
3. $h(x) = \frac{1}{x^2+1}$.

Discontinuous functions

To show from the (ε, δ)-definition of continuity that a function is discontinuous at a point x_0, we need to negate the statement: "For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$."
Its negative is the following (check that you understand this!):
"There exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point x with $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$."

Example 3. *Question: Let $g : \mathbb{R} \to \mathbb{R}$ be defined by*

$$g(x) = \begin{cases} x^2 - 3 & \textit{if } x \leq 0, \\ \sin(x) & \textit{if } x > 0. \end{cases}$$

Prove that g is discontinuous at 0.
Answer: ***Let*** $\varepsilon = 2$ ***and let*** $\delta > 0$ ***be given. Choose*** $x \in (0, \delta)$ *with $x < \pi$. Then $g(x) = \sin(x) > 0$. Hence,*

$$|x - 0| < \delta \quad \textit{and} \quad |g(x) - g(0)| = |\sin(x) - 3| > 3 \geq \varepsilon$$

and g is discontinuous at 0.
Notes:

1. *The wording in bold always appears in the proof of discontinuity using the (ε, δ)-definition.*
2. *It's probably good to sketch the graph of the function.*
3. *Think about how big the gap is. This determines which ε we can choose, here we could have chosen anything up to 3.*
4. *You will need to choose x, so that it's in a suitable range (here we chose $0 < x < \pi$).*

Exercises 4. *Prove that the following functions are discontinuous at $x = 1$.*

$$f(x) = \begin{cases} x^2 - 3 & \textit{if } x < 1, \\ x - 1 & \textit{if } x \geq 1, \end{cases} \qquad \textit{and} \qquad g(x) = \begin{cases} 1/(x-1) & \textit{if } x \neq 1, \\ 0 & \textit{if } x = 1. \end{cases}$$