

MA552 Analysis Worksheet 3

Convergence of series - Part 1

We consider the convergence of a series $\sum a_j$ to be the convergence of the sequence of partial sums,

$$s_n = \sum_{j=1}^n a_j.$$

If the sequence $\{s_n\}$ converges, say $s_n \rightarrow \ell$ then we write

$$\sum_{j=1}^{\infty} a_j = \ell.$$

By and large, we can prove only that ℓ exists. It is relatively rare that we know exactly what ℓ is as a real number.

There are 2 families for which we know convergence.

- We have

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and otherwise diverges. This is proved by comparing $\int_1^N x^{-p} dx$ with its “upper and lower sums”; in the textbook this is called the integral test. More practice with the integral test:

Q1. Prove

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for $p = 19/4$ converges, by comparing it with the integral,

$$\int_1^N \frac{1}{x^p} dx, \quad p = 19/4.$$

Q2. By comparing

$$\sum_{m=2}^n \frac{1}{m \log m}$$

with the integral

$$\int_2^n \frac{1}{x \log x} dx$$

determine whether the series converges or diverges.

- We have the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad -1 < r < 1$$

and diverges if $|r| \geq 1$. By differentiating

$$\sum_{n=0}^N r^n$$

term by term with respect to r , one can obtain both convergence, and the limit, for series of the form $\sum_{n=1}^{\infty} n^p r^n$ where $|r| < 1$ and $p \in \mathbb{N}$. • Find $\sum_1^{\infty} n^p / 2^n$ for $p = 1, 2, 3, 4, 5$.

Q3. Determine all real values of x for which each of the following series converges and find the sum to infinity.

$$\begin{array}{ll} \text{(i)} & \sum_{n=0}^{\infty} \frac{x^n}{3^n} & \text{(ii)} & \sum_{n=0}^{\infty} \frac{n3^n}{x^n} \\ \text{(iii)} & \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} & \text{(iv)} & \sum_{n=0}^{\infty} \frac{2^n}{3^n} \left(x - \frac{1}{x}\right)^n \end{array}$$

From these two families, the convergence of many other series can be deduced using the *comparison test* and the *limit form of the comparison test*. These are used on series with all positive terms, so if you are comparing with a geometric series, you need $0 < r < 1$.

To use these tests, you need to make an educated guess as to which series you are going to compare the series you are given with. To use the comparison test, your skills in proving inequalities are needed. For the limit form of the comparison test, your skills in finding limits are needed.

Q4. Show the convergence or divergence of the following series.

$$\begin{array}{ll} \text{(i)} & \sum \frac{1}{\log 2n} & \text{(ii)} & \sum \frac{2^n + 3^n}{3^n + 4^n} \\ \text{(iii)} & \sum \sin\left(\frac{1}{n^2}\right) & \text{(iv)} & \sum \frac{2^n}{n^2 + 3^n} \\ \text{(v)} & \sum \frac{n}{\sqrt{2n^3 + 1}} & \text{(vi)} & \sum \frac{1}{3n - 2} \\ \text{(vii)} & \sum \frac{n + 1}{n^3 + 1} & \text{(viii)} & \sum \frac{n}{2n - 1} \\ \text{(ix)} & \sum \log\left(1 + \frac{1}{n}\right) & \text{(x)} & \sum \frac{1}{(\log n)^n} \end{array}$$

Prof. Elizabeth Mansfield