MA552 Analysis Worksheet 2

Convergence of sequences using theorems.

We have discussed the following theorems and examples (or examples like these) in lectures. Use this sheet as a way to make sure you have covered everything in your revision.
All proofs of convergence of sequences from theorems must state the name of the theorem, and demonstrate that the hypotheses of the theorem hold.

1. *Algebra of Limits Theorem.* Sample question and model answer:

   Question Show that

   $$\frac{2n^2+3n+1}{5n^2-2n+40} \to \frac{2}{5}$$

   Solution

   $$\begin{aligned} \lim \frac{2n^2+3n+1}{5n^2-2n+40} &= \lim \frac{2+3\frac{1}{n}+\frac{1}{n^2}}{5-2\frac{1}{n}+40\frac{1}{n^2}} \\ &= \frac{2+3\lim\frac{1}{n}+\lim\frac{1}{n^2}}{5-2\lim\frac{1}{n}+40\lim\frac{1}{n^2}} \\ &= \tfrac{2}{5} \end{aligned}$$

   by the Algebra of Limits theorem, since all limits exist and the denominator is not zero, including in the limit.

   Prove the following sequences have the given limits.

   $$\text{(i)} \quad \frac{2n^7+9n-3}{8n^7+6n^3+5} \to \frac{1}{4}$$

   $$\text{(ii)} \quad \frac{3^n+5^n}{3^{n+1}+5^{n+1}} \to \frac{1}{5}$$

   For (ii), you will need to state that $c^n \to 0$ for $|c| < 1$.

   If $x_n \to \ell$, $x_n > 0$ for all n, $x_1 = \sqrt{2}$ and $x_{n+1}^2 = 2 + x_n$, show $\ell = 2$. Does the result change if $x_1 = 300$?

2. *An increasing sequence bounded above converges. A decreasing sequence bounded below converges.*

   (a) Show

   $$\frac{1}{5^n} \to 0$$

   (b) Show the sequence

   $$\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

   converges to 2.

3. *Sandwich Theorem*: Show the following

(i) if $x_n \to 0$ then

$$\frac{\sin x_n}{x_n} \to 1$$

(ii)

$$n \log\left(\frac{n+1}{n}\right) \to 1$$

(iii)

$$c^{1/n} \to 1, \qquad c > 0$$

(iv)

$$n^{1/n} \to 1$$

(v)

$$\frac{100^n}{n!} \to 0$$

(vi)

$$\sqrt{n+4} - \sqrt{n} \to 0$$

(vii)

$$\frac{n^2}{2^n} \to 0$$

These can all be combined with the Algebra of Limits theorem. Show $n^3/2^n \to 0$ and hence show that

$$\frac{2^n + n^2}{2^{n+1} + n^3} \to \frac{1}{2}.$$

4. *Other results*:

- If $x_n \to \ell$ then $\exp x_n \to \exp \ell$
- If $x_n \to \ell$ then

$$z_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \to \ell$$

Show that

(a)

$$\left(1 + \frac{1}{n}\right)^n \to e$$

(b)

$$\frac{(n!)^{1/n}}{n} \to \frac{1}{e}$$

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