# Mathematical Analysis Worksheet 11

## Riemann integration

The aim of Riemann integration is to determine the (signed) area between the graph of a function $f$ and the $x$-axis in an interval $[a, b]$. Given a partition of the interval $[a, b]$

$$\mathcal{P} : a = t_0 < t_1 < t_2 < ... < t_n = b,$$

boundedness of the function $f$ is sufficient to show that

$$M_i := \sup\{f(x) : x \in [t_{i-1}, t_i]\} \quad \text{and} \quad m_i := \inf\{f(x) : x \in [t_{i-1}, t_i]\}$$

exist for $i = 1, ..., n$. This allows us to define the *upper and lower Riemann sum* (with respect to the given partition) as

$$U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \quad \text{and} \quad L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

When asked to determine upper and lower sums it is usually best to sketch the function and the partition, even if it's not required in the question. However, I will not do this on this worksheet.

**Example 1.** *Question: Let $f(x) = \sin(x)$ and $\mathcal{P}$ be the partition $0 < \pi/6 < \pi/4 < \pi/3 < 2\pi/3 < \pi$. Determine the upper and lower sums of $f$ with respect to $\mathcal{P}$.*
*Answer: We need to first determine the $M_i$ and $m_i$. As $f$ is increasing on the first three subintervals $[0, \pi/6]$, $[\pi/6, \pi/4]$ and $[\pi/4, \pi/3]$, we have*

$$m_1 = f(t_0) = \sin(0) = 0, \quad M_1 = f(t_1) = \sin(\pi/6) = 1/2, \quad m_2 = f(t_1) = \sin(\pi/6) = 1/2,$$

$$M_2 = f(t_2) = \sin(\pi/4) = \frac{1}{\sqrt{2}}, \quad m_3 = f(t_2) = \sin(\pi/4) = \frac{1}{\sqrt{2}}, \quad M_3 = f(t_3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}.$$

*On the fourth subinterval from $\pi/3$ to $2\pi/3$ the function takes its maximum at the midpoint and its minimum at either of the endpoints:*

$$m_4 = f(t_3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad M_4 = \sin(\pi/2) = 1.$$

*On the final subinterval $[2\pi/3, \pi]$, $f$ is monotonically decreasing, so*

$$m_5 = f(t_5) = \sin(\pi) = 0, \quad M_5 = f(t_4) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}.$$

*Hence,*

$$\begin{aligned} L(f, \mathcal{P}) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= 0 \cdot \left(\frac{\pi}{6} - 0\right) + \frac{1}{2} \cdot \left(\frac{\pi}{4} - \frac{\pi}{6}\right) + \frac{1}{\sqrt{2}} \cdot \left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{\sqrt{3}}{2} \cdot \left(\frac{2\pi}{3} - \frac{\pi}{3}\right) + 0 \cdot \left(\pi - \frac{2\pi}{3}\right) \\ &= \pi \left(\frac{1}{24} + \frac{1}{12\sqrt{2}} + \frac{1}{2\sqrt{3}}\right), \end{aligned}$$

$$
\begin{aligned}
U(f,\mathcal{P}) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \\
&= \frac{1}{2}\cdot\left(\frac{\pi}{6}-0\right)+\frac{1}{\sqrt{2}}\cdot\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}\cdot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+1\cdot\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\cdot\left(\pi-\frac{2\pi}{3}\right) \\
&= \pi\left(\frac{5}{12}+\frac{1}{12\sqrt{2}}+\frac{5}{8\sqrt{3}}\right).
\end{aligned}
$$

Remember that not all bounded functions are integrable (e.g. $\chi_{\mathbb{Q}}$, the characteristic function of the rationals, on $[0,1]$). There are several useful criteria for integrability. The Riemann criterion is most often used in proofs. A simple sufficient criterion for integrability is continuity. If we already know from theoretical considerations that a function is integrable, it makes it easier to calculate the integral:

**Example 2.** *Question (2009 exam): Construct $L(f,\mathcal{P}_n)$ and $U(f,\mathcal{P}_n)$ for $f(x)=\frac{1}{x}$ on $[1,2]$ for the partition*

$$\mathcal{P}_n : 1 < 1+\frac{1}{n} < 1+\frac{2}{n} < \ldots < 1+\frac{n}{n} = 2.$$

*Let*

$$x_n = \sum_{j=1}^{2n}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j}.$$

*Show that*

$$x_n = L(f,\mathcal{P}_n) < \log(2) < x_n + \frac{1}{2n} = U(f,\mathcal{P}_n).$$

*Hence, or otherwise, prove $x_n \to \log(2)$.*
*Answer: As the function is monotonically decreasing, we have*

$$L(f,\mathcal{P}) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) = \sum_{i=1}^{n}\frac{1}{1+i/n}\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{1}{n+i}$$

*and*

$$
\begin{aligned}
U(f,\mathcal{P}) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) = \sum_{i=1}^{n}\frac{1}{1+(i-1)/n}\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n}\frac{1}{n+i-1} = \sum_{i=0}^{n-1}\frac{1}{n+i} = \sum_{i=1}^{n}\frac{1}{n+i}+\frac{1}{n}-\frac{1}{2n} = \sum_{i=1}^{n}\frac{1}{n+i}+\frac{1}{2n}.
\end{aligned}
$$

*Note that*

$$x_n = \sum_{j=1}^{2n}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j} = \sum_{j=1}^{n}\frac{1}{n+j}$$

*and that $f$ is integrable (as it's continuous) with* $\log(2) = \int_1^2 \frac{1}{x}\, dx$ *(by definition of the* log*-function). As the integral lies between the upper and the lower sum, we obtain*

$$x_n = L(f, \mathcal{P}_n) < \log(2) < U(f, \mathcal{P}_n) = x_n + \frac{1}{2n}.$$

*As $f$ is integrable and $\|\mathcal{P}_n\| \to 0$ as $n \to \infty$, we have* $\lim_{n\to\infty} L(f, \mathcal{P}_n) = \int_1^2 \frac{1}{x}\, dx = \lim_{n\to\infty} U(f, \mathcal{P}_n)$. *Therefore, $x_n \to \log(2)$.*

Particularly useful in integrating functions is the
**Fundamental Theorem of Calculus**: Let $f$ be Riemann integrable on $[a, b]$ and $F$ a primitive for $f$ on $[a, b]$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

**Example 3.** *Question: Determine* $\int_0^{\pi/2} \frac{1}{1+x^2}\, dx$.
*Answer: Let* $f(x) = \frac{1}{1+x^2}$. *Then $f$ is continuous on $[0, \pi/2]$ and so it is integrable. Moreover, as* $F(x) = \arctan(x)$ *satisfies $F'(x) = f(x)$, it is a primitive for $f$. Hence, by the Fundamental Theorem of Calculus*

$$\int_0^{\pi/2} \frac{1}{1+x^2}\, dx = \arctan(\pi/2) - \arctan(0) = 1.$$

**Exercises 4.**

1. *On $[-1, 3]$, let $f(x) = |x - 2| + 1$ and let $\mathcal{P}$ be the partition*

$$\mathcal{P}: \quad -1 < 0 < \frac{1}{2} < 1 < 3.$$

   *Sketch the graph of the function $f$ and indicate the rectangles whose areas contribute to the upper and lower Riemann sums. Determine the lower Riemann sum $L(f, \mathcal{P})$ and the upper Riemann sum $U(f, \mathcal{P})$.*

2. *Let* $f(x) = \sin\left(\frac{\pi x}{2}\right)$ *and let $\mathcal{P}_n$ be the partition of $[0, 1]$ given by*

$$\mathcal{P}_n : 0 < \frac{1}{n} < \frac{2}{n} < ... < \frac{n-1}{n} < 1.$$

   *Show that*

$$L(f, \mathcal{P}_n) = \sum_{j=0}^{n-1} \frac{1}{n} \sin\left(\frac{j\pi}{2n}\right).$$

   *Hence, show*

$$\frac{1}{n}\left(\sin\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right) + ... + \sin\left(\frac{(n-1)\pi}{2n}\right)\right) \to \frac{2}{\pi}.$$