Mathematical Analysis Worksheet 10

Power Series & Taylor Series

A *power series* is a series of the form $\sum_{k=0}^{\infty} a_k x^k$, where the a_k are real numbers and x is a variable. We think of a power series as a function defined for those x for which the series converges. It is important to know for which x a power series converges. There are only three possiblities:

- The series converges for all $x \in \mathbb{R}$. In this case we say the radius of convergence R is ∞.
- The series diverges for all $x \neq 0$. In this case we say the radius of convergence R is 0.
- There exists $R \in \mathbb{R}$ such that the series converges for $|x| < R$ and diverges for $|x| > R$. In this case R is called the radius of convergence.

The question of determining those x for which a power series converges therefore reduces to the question of determining the radius of convergence, and if R is finite and non-zero, checking the cases $x = \pm R$ separately.
Usually, the easiest way of determining the radius of convergence is by using the ratio test.

Example 1. *Question: Determine the radius of convergence of the power series* $\sum_{k=0}^{\infty} 2^k x^k$
Answer: Let $b_k = 2^k x^k$. *Then*

$$\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{2^{k+1}x^{k+1}}{2^k x^k}\right| = 2|x|.$$

By the ratio test, the series converges for $|x| < 1/2$ *and diverges for* $|x| > 1/2$. *Its radius of convergence is* $1/2$.

If the question is formulated differently, we may need to check the cases $x = \pm R$.

Example 2. *Question: Determine those* $x \in \mathbb{R}$ *for which the power series* $\sum_{k=1}^{\infty} \frac{x^k}{k}$ *converges.*
Answer: We first determine the radius of convergence of the power series. Let $b_k = x^k/k$. *Then*

$$\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{kx^{k+1}}{(k+1)x^k}\right| = \frac{k}{k+1}|x| \to |x| \text{ as } k \to \infty.$$

By the ratio test, the series converges for $|x| < 1$ *and diverges for* $|x| > 1$. *For* $x = 1$, *the series equals* $\sum_{k=1}^{\infty} \frac{1}{k}$, *the harmonic series which diverges, while for* $x = -1$, *the series equals* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, *the alternating harmonic series which converges by the Alternating Series Test. Hence, the series converges for* $x \in [-1, 1)$ *and diverges for all other* x.

Within its radius of convergence, a power series can be differentiated (and integrated) term by term, e.g. given $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$, we have

$$f'(x) = \sum_{k=1}^{\infty} x^{k-1} \text{ and } \int_0^x f(t)dt = \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} \text{ for } |x| < 1.$$

Exercises 3. 1. *Determine the radius of convergence of the following power series:*

$$(i) \sum_{k=1}^{\infty} \left(\frac{x}{4}\right)^k, \quad (ii) \sum_{k=1}^{\infty} \frac{k^2 x^k}{2^k}, \quad (iii) \sum_{k=1}^{\infty} \frac{k! x^k}{k^k}.$$

2. Determine those $x \in \mathbb{R}$ for which the following power series converge.

$$\textit{(i)} \sum_{k=1}^{\infty} (5x)^k, \quad \textit{(ii)} \sum_{k=1}^{\infty} \frac{(5x)^k}{k}, \quad \textit{(iii)} \sum_{k=1}^{\infty} \frac{(5x)^k}{k^2}.$$

Due to these properties of power series, it can be useful to represent a function as a power series. The *Taylor series* of a function f at $x = a$ is given by the (formal!) series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

In practice, a function is not always infinitely differentiable, so the expression may not make sense. Moreover, one may not be able to compute the infinitely many terms of the series. Therefore one is interested in approximating the function f in a neighbourhhod of the point a by the first few terms of the series, i.e. a polynomial, and estimating the error made. This can be done using **Taylor's Theorem**: Suppose that $f, f', ..., f^{(n)}$ are continuous on $[a, x]$ and $f^{(n+1)}$ exists on (a, x). Then there exists $c \in (a, x)$ such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{Taylor polynomial of degree n}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{error term / remainder}} .$$

Example 4. *Question (2009 exam): Apply Taylor's Theorem to $f(x) = e^{\sin(x)}$ with $a = 0$ and $n = 2$ to give a polynomial approximation of degree 2 for f and clearly indicate the error term.*
Answer: We first note that as a composition of the continuous and infinitely differentiable functions exp *and* sin*, the function f is continuous and infinitely differentiable. In particular, it is three times differentiable and by the chain and product rules*

$$f'(x) = \cos(x)e^{\sin(x)}, \quad f''(x) = -\sin(x)e^{\sin(x)} + \cos^2(x)e^{\sin(x)},$$

$$f'''(x) = -\cos(x)e^{\sin(x)} - 3\sin(x)\cos(x)e^{\sin(x)} + \cos^3(x)e^{\sin(x)}.$$

Hence, $f(0) = f'(0) = f''(0) = 1$ and the polynomial approximation of degree 2 at $a = 0$ is given by

$$p(x) = \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k = 1 + x + \frac{1}{2}x^2.$$

The error term is given by

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = \frac{-\cos(c)e^{\sin(c)} - 3\sin(c)\cos(c)e^{\sin(c)} + \cos^3(c)e^{\sin(c)}}{6}x^3$$

for some $c \in (0, x)$.

To get a bound on the error, we have to find (or give a good estimate on) the maximum of the modulus of the error term on the relevant interval.

Example 5. *Question: Apply Taylor's Theorem to $f(x) = \sin(x)$ with $a = 0$ and $n = 2$ to give a polynomial approximation of degree 2 for f on the interval $(-\pi/2, \pi/2)$ and give a bound on the error term.*

Answer: We first note that f is continuous and infinitely differentiable and

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x).$$

Hence, $f(0) = f''(0) = 0$ and $f'(0) = 1$ and the polynomial approximation of degree 2 at $a = 0$ is given by

$$p(x) = \sum_{k=0}^{2} \frac{f^{(k)}(a)}{k!}(x-a)^k = 0 + x + 0x^2 = x.$$

The error term is given by

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = \frac{-\cos(c)}{6}x^3$$

for some $c \in (0, x)$. As we want to approximate f on the interval $(-\pi/2, \pi/2)$, this is the region in which the x varies and so also does c. The maximum of the modulus of $-cos(c)$ is achieved at $c = 0$. Hence the error can be bounded by

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| = \left|\frac{-\cos(c)}{6}x^3\right| \leq \frac{1}{6}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{48}.$$

Note that the error in the last example is pretty large, so it's not a particularly good idea to approximate $\sin(x)$ by x on the whole of the interval $(-\pi/2, \pi/2)$.

Exercises 6. *1. Use Taylor's Theorem to find polynomial approximations of degree 3 for the given functions at the given point. In each case, state the error term.*

$$\textit{(i)}\ \sin(e^x),\ a = 1, \qquad \textit{(ii)}\ \frac{1}{1+x^2},\ a = 0, \qquad \textit{(iii)}\ \arctan(x),\ a = 0.$$

2. Use Taylor's Theorem to find polynomial approximations of degree 3 at $a = 0$ for the given functions on the interval $(-1/2, 1/2)$. In each case, give a bound on the error term.

$$\textit{(i)}\ x\sin(x), \qquad \textit{(ii)}\ \tan(x), \qquad \textit{(iii)}\ \exp(e^x).$$