MA552 Analysis Worksheet 1

Convergence of sequences using the definition.

Recall the definition:
We say $x_n \to \ell$ if for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$n > N \Longrightarrow |x_n - \ell| < \varepsilon.$$

All proofs of convergence of sequences from the definition follow the same format, which shows the requirements of the definition hold. Here is the first of two examples.

1. *Question*: Prove that $\dfrac{n+1}{n} \to 1$.

   Answer:

   Let $\varepsilon > 0$ **be given.**

   Let $N = N(\varepsilon)$ **be an integer greater than** $1/\varepsilon$.

   Then for all $n > N$ **we have**

$$\begin{aligned} |\boldsymbol{x_n} - \boldsymbol{\ell}| &= \left|\frac{n+1}{n} - 1\right| \\ &= \frac{1}{n} \\ &< \frac{1}{N} \qquad \text{as } n > N \\ &< \boldsymbol{\varepsilon} \qquad \textbf{as required}. \end{aligned}$$

Notes:

1. The wording in bold always appears if “Prove” appears in the question.
2. The sequence of inequalities ending in “$< \varepsilon$ as required” depends on the sequence being studied. This is where your skill and experience come in.
3. The value of $N(\varepsilon)$ is filled in last, when you know what it has to be from your sequence of inequalities ending in “$< \varepsilon$ as required”.

Prove the following sequences have the given limits.

$$(i) \quad x_n = \frac{n+1}{2n+3}, \qquad \ell = \frac{1}{2}$$

$$(ii) \quad x_n = \frac{1}{n} + (-1)^n \frac{1}{n^2}, \qquad \ell = 0$$

$$(iii) \quad x_n = \sqrt{n+3} - \sqrt{n+1}, \qquad \ell = 0$$

$$(iv) \quad x_n = \frac{\sin n}{n^2}, \qquad \ell = 0$$

In this second example, we show that there may be more than one condition on $N(\varepsilon)$.

2. *Question*: Prove that $\dfrac{n+10^6}{n^2} \to 0$.

   Answer:

   Let $\varepsilon > 0$ **be given.**

   Let $N = N(\varepsilon)$ **be an integer greater than** $\max\{10^6, 2/\varepsilon\}$.

   Then for all $n > N$ **we have**

$$\begin{aligned}
|\boldsymbol{x_n} - \boldsymbol{\ell}| &= \left|\frac{n+10^6}{n^2}\right| \\
&< \frac{2n}{n^2} \qquad \text{as } n + 10^6 < 2n \text{ since } n > N > 10^6 \\
&< \frac{2}{N} \qquad \text{as } n > N \\
&< \boldsymbol{\varepsilon} \qquad \textbf{as required}.
\end{aligned}$$

Prove the following sequences have the given limits.

$$\begin{aligned}
&(i) \quad x_n = \frac{2n+10^8}{n^2}, \qquad && \ell = 0 \\
&(ii) \quad x_n = \frac{3n^2+100n+4000}{2n^2+3}, \qquad && \ell = \frac{3}{2}
\end{aligned}$$

Question: An engineer works to a tolerance of $\varepsilon = 10^{-6}$. She knows the process she is modelling involves a sequence

$$x_n = \begin{cases} \dfrac{2}{n^2+3} & n \text{ is even} \\ \text{unknown} & n \text{ is odd} \end{cases}$$

but such that $0 < x_{n+1} < x_n$. How big must N be in order that

$$|x_n| < 10^{-6}, \qquad \text{for all } n > N.$$

Can you answer the question if she knows only that $0 < x_{n+1} < x_{n-1}$?

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