Mathematical Analysis (MA552)

Exercise Sheet Four

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x}\sin^2(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Show that f is continuous at $x = 0$.

   (b) Show that f is differentiable at $x = 0$.

   (c) Determine $f'(x)$ for all $x \neq 0$.

   (d) Is f' continuous at 0?

   Hint: L'Hôpital's rule may come in handy for this question.

2. The aim of this question is to use the definition of the derivative to show that the function $f(x) = \log(x)$ is differentiable for all $x > 0$ and that $f'(x) = \frac{1}{x}$.

   (a) First show that for $c > 0$ and $c + h > 0$

$$\frac{f(c+h) - f(c)}{h} = \frac{1}{h}\int_c^{c+h} \frac{\mathrm{d}t}{t}.$$

   (You may use all properties of the integral that you know from a Calculus course.)

   (b) By considering the area below the graph of f show that

$$\frac{h}{c+h} \leq \int_c^{c+h} \frac{\mathrm{d}t}{t} \leq \frac{h}{c} \quad \text{for} \quad h > 0.$$

   Hint: Treat the cases $h > 0$ and $h < 0$ separately.

   (c) Use the Sandwich Rule to determine

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

   Hint: Start by again treating the cases $h > 0$ and $h < 0$ separately.

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3. (a) Determine the following limits:

$$\text{(i) } \lim_{x\to 0} \frac{\tan(x) - x}{x - \sin(x)}, \qquad \text{(ii) } \lim_{x\to 0} x\cos\left(\frac{1}{x}\right), \qquad \text{(iii) } \lim_{x\to 0} \frac{\log(1+tx)}{x} \text{ for } t \in \mathbb{R}.$$

(b) Use the limit in (iii) with $t = 1$ and a suitable sequence (x_n) to show that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

4. (a) Prove that if $f'(x) \geq M$ for all $x \in (a, b)$, then $f(b) \geq f(a) + M(b - a)$.

(b) Using results from the lectures, find all functions f such that

i. $f'(x) = \sin(x)$,

ii. $f''(x) = x^2$,

iii. $f(x) = 2f'(x)$.

5. Find the radius of convergence for the following power series:

$$\text{(i) } \sum_{k=1}^{\infty} \frac{x^k}{(2k)!}, \qquad \text{(ii) } \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad \text{(iii) } \sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k.$$

6. (a) Let $\alpha \in \mathbb{R}$. Find the Taylor series expansion for $f(x) = (1+x)^\alpha$ at $x = 0$.

(b) Use the first four terms of the series in (a) (i.e. the cubic Taylor polynomial) to give an estimate of $\left(\frac{3}{2}\right)^{1/3}$.

(c) Use the Lagrange form of the remainder to show that the error in (b) is less than $2.6 \cdot 10^{-3}$.