MA552 Analysis

Exercise Sheet Two

Q1. Show the following limits. You may use the theorems proved in lectures, but you must make it clear that any hypotheses, required by the theorems you use, do in fact hold.

(i) $\dfrac{3n^2 - 5n - 7}{n^2 + 6n + 1} \to 3$

(ii) $\dfrac{5n + (-1)^n}{2n + 17\sqrt{n}} \to \dfrac{5}{2}$

(iii) $\dfrac{7^{n+1} + 10^{n+1}}{7^n + 10^n} \to 10$

(iv) $\dfrac{7^{n+1} + n!}{7^n + 10(n!)} \to \dfrac{1}{10}$

(v) Consider

$$x_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

where the dot denotes multiplication. By noting that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

or otherwise, show the sequence converges and find the limit.

(vi) $\left(1 + \dfrac{1}{n!}\right)^{n!} \to e$

(vii) $\left(1 - \dfrac{1}{n}\right)^n \to e^{-1}$

(viii) $\sqrt{n+1} - \sqrt{n} \to 0$

Q2. Show that

$$\frac{n^2}{3^n} \to 0, \qquad \frac{n^2 2^n}{5^n} \to 0.$$

More generally, show that for $-1 < x < 1$,

$$n^3 x^n \to 0.$$

Hint: use the binomial theorem, for example, $3^n = (1+2)^n$.

Q3. Consider the sequence,

$$x_1 = \sqrt{2}, \quad x_2 = \sqrt{2\sqrt{2}}, \quad x_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \quad \cdots$$

By formulating x_{n+1} as a function of x_n, show that the sequence is increasing and bounded above by 2. Conclude that the sequence converges and find the limit.

Q4. Suppose that $x_n \to \ell$ and that $x_n \geq 0$ for all n. Show that

(i) $\ell \geq 0$

(ii) $\sqrt{x_n} \to \sqrt{\ell}$.

Hint If $\ell \neq 0$,

$$|\sqrt{x_n} - \sqrt{\ell}| = \frac{|x_n - \ell|}{|\sqrt{x_n} + \sqrt{\ell}|} < \frac{|x_n - \ell|}{\sqrt{\ell}}.$$

Use these results to show that

(i) $\dfrac{\sqrt{3n^2+1}}{2n-1} \to \dfrac{\sqrt{3}}{2}$

(ii) $(\sqrt{n+1} - \sqrt{n})\sqrt{n+\frac{1}{2}} \to \frac{1}{2}$

(iii) $\sqrt{n^2+n} - n \to \frac{1}{2}$

Q5. Consider the sequence,

$$x_n = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}$$

By comparing x_n with a suitable product of the smallest term in its sum, show that the sequence diverges to infinity, that is, increases without bound. By contrast, consider the sequence,

$$x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}.$$

By comparing x_n with (suitable products of) each of the highest and smallest terms in its sum, show that the limit of the sequence is 1.

Q6. *Challenge* If $x_n > 0$ and $y_n = x_{n+1}/x_n \to \ell$, show that

$$x_n^{1/n} \to \ell$$

You may assume it is enough to show that $\log(x_n)/n \to \log(\ell)$.
Hint: Simplify $y_1 y_2 \cdots y_n$.
Use this result to find the limits of

$$n^{1/n}, \qquad (n^5 + n^4)^{1/n}, \qquad \left(\frac{n!}{n^n}\right)^{1/n}$$