MA552 Analysis

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Exercise Sheet One

Q1. *Review of mathematical induction* Consider the Newton-Raphson sequence for $\sqrt{2}$ given in lectures, which is,

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

Show, using the method of mathematical induction, that

$$\text{(a)} \quad x_n > 0 \qquad \text{(b)} \quad x_n > \sqrt{2}, \qquad \text{(c)} \quad x_n - x_{n+1} > 0$$

for all n. In proving (b) you will need (a), and in proving (c) you will need both (a) and (b). Thus, you have shown that

$$\sqrt{2} < x_{n+1} < x_n.$$

Does this, by itself, prove that $x_n \to \sqrt{2}$ as $n \to \infty$?

Hint for (b): factor

$$\frac{x_k}{2} + \frac{1}{x_k} - \sqrt{2}.$$

Q2. *Review of absolute value* Recall the definition of absolute value: $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Graph

$$\text{(i) } y = |x/2 + 1| \quad \text{(ii) } y = |x^2/2 - 1| \quad \text{(iii) } y = |2x + 1| + |2x - 1|$$

Recall the triangle inequality,

$$||a| - |b|| < |a + b| < |a| + |b|$$

which is valid for all real numbers a and b. Show, by drawing three graphs in the same plot,

$$||x + 2| - |x - 2|| \leq |(x + 2) + (x - 2)| \leq |x + 2| + |x - 2|$$

for all x.

Q3. *Review of inequalities* Recall

$$0 < a < b \Longrightarrow \frac{1}{a} > \frac{1}{b}$$

Example calculation: First note that

$$\begin{aligned} n^3 + 57n - 3 &< n^3 + 57n && \text{all } n \\ &< 2n^3 && \text{all } n > 7 \end{aligned}$$

and also

$$\begin{aligned} n^3 - 19n + 7 &> n^3 - 19n && \text{all } n \\ &> n^3/2 && \text{all } n > 6 \end{aligned}$$

Also, both expressions are positive for $n > 1$. Hence

$$\left|\frac{n^3 + 57n - 3}{n^3 - 19n + 7}\right| < \frac{2n^3}{n^3} = 2 \qquad \text{all } n > 7$$

Using the above as a template, fill in the missing information

(i)

$$\left|\frac{2n^3 - 5n}{5n^4 + 9}\right| <?$$

(ii)

$$\left|\frac{n + 5}{4n^2 - n + 2}\right| <?$$

Find $c > 0$ and $N \in \mathbb{N}$ such that

$$\left|\frac{5n^4 + 19}{n^3 + 91n + 35}\right| > cn$$

for $n > N$.

Q4. *Review of the binomial theorem* Let $\binom{n}{r}$ denote the binomial coefficient $\dfrac{n!}{r!(n-r)!}$.

(i) Show that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$.

(ii) Use $(a+b)^n(a+b) = (a+b)^{n+1}$ and mathematical induction to show that

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

(iii) Show that

$$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

(iv) Show that if $y > 0$ then both

$$(1+y)^n > ny \qquad \text{and} \qquad (1+y)^n > \frac{n(n-1)}{2} y^2$$

(v) Show that

$$0 < \frac{1}{2^n} < \frac{1}{n}.$$