Mathematical Analysis (MA552)

Assignment 3

This assignment counts for 5% of the final mark. Full marks will be awarded for 100 marks. You may use any theorems proved in lectures or in the exercise sheets, but you must tell me which theorems you are using, and why the hypotheses they require hold.

DUE DATE Noon, Friday 14th May, 2010

1. (a) State without proof the Mean Value Theorem, specifying all conditions required of the function concerned to guarantee the conclusion of the theorem. [5 marks]

   (b) A function $f : [0, \infty) \to \mathbb{R}$ satisfies the following conditions:

   (I) f is continuous and $f(0) = 0$,

   (II) f is differentiable, $f'(x) > 0$ and $f(x) \le f'(x)^2$ for all $x > 0$.

   Show that this implies the following:

   i. $f(x) > 0$ for all $x > 0$. [6 marks]

   ii. By considering $g(x) = \sqrt{f(x)}$, show that $f(x) \ge \frac{1}{4}x^2$. [9 marks]

2. Let f be a real valued function of a real variable x. Suppose that f and its first n derivatives are continuous on $[a, x]$ and differentiable on (a, x).

   (a) State without proof Taylor's Theorem giving a polynomial approximation of degree n for $f(x)$. [6 marks]

   (b) Use the Taylor polynomial of degree 3 at $a = 1$ of the function $f(x) = \sin(\pi x)$ to give an approximation to $\sin(3)$. [8 marks]

   (c) Show that the error in (b) is less than $2 \cdot 10^{-5}$. (You are expected to use the error term from Taylor's Theorem here.) [6 marks]

3. Given a bounded function $f : [a, b] \to \mathbb{R}$, and a partition

$$\mathcal{P} : a = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_n = b,$$

   define the *lower Riemann sum* $L(f, \mathcal{P})$ and the *upper Riemann sum* $U(f, \mathcal{P})$ of f with respect to $\mathcal{P}$. [7 marks]

   (a) Determine the lower Riemann sum for $f(x) = 1/x$ on $[1, 2]$ with respect to the partition $\mathcal{P}_n$ given by $1 = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_n = 2$ with $\xi_i = 1 + \frac{i}{n}$ for $i = 0, \dots, n$ and $n \in \mathbb{N}$. [6 marks]

(b) Use this result to show

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \ln 2.$$

[7 marks]

4. For a set $A \subseteq \mathbb{R}$ we define its characteristic function by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \;\; \text{otherwise.} \end{cases}$$

For the following sets A, determine whether χ_A is Riemann integrable over $[0,1]$ and calculate $\int_0^1 \chi_A(x)dx$ when possible.

(a) $A = [0, \frac{1}{2}]$. [13 marks]

(b) $A = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. [10 marks]

(c) $A = [0,1] \setminus \mathbb{Q}$. [7 marks]

(d)* $A = \{\frac{1}{k} : k \in \mathbb{N}\}$. [20 marks]

Hint: In most cases, it is simplest to consider partitions

$$\mathcal{P}_n : 0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n-1} < 1,$$

and use the Riemann criterion (Theorem 17).

1) a) Let f be continuous (1) on $[a,b]$ and differentiable (1) on (a,b). Then there exists $c \in (a,b)$ (1) such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$ (2)

b) i) We apply the MVT to f on $[0,x]$ for any $x>0$ (1).
As f is continuous on $[0,x]$ and differentiable on $(0,x)$ (1), there exists $c \in (0,x)$ such that
$$f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}.$$ (2)

Now, $f'(c) > 0$, so $\frac{f(x)}{x} > 0$. As $x > 0$, this implies $f(x) > 0$. (2)

ii) As $f(x) \geq 0$ for all $x \in [0,\infty)$, g is continuous on $[0,x]$ and differentiable on $(0,x)$ for all $x>0$. (2)

By the MVT, there exists $c \in (0,x)$ such that
$$g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$$ (2)

Moreover, by the chain rule,

$$g'(c) = \frac{f'(c)}{2\sqrt{f(c)}} \qquad 2$$

and by the assumptions on f, we have

$$g'(c) \geq \frac{f'(c)}{2\sqrt{f'(c)^2}} = \frac{1}{2} \qquad (\text{as } f'(c) > 0)$$

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Therefore $\frac{g(x)}{x} \geq \frac{1}{2}$ and

$$\frac{g(x)}{x} = \frac{\sqrt{f(x)}}{x} \geq \frac{1}{2}$$ implies (as $f(x) \geq 0$)

$$f(x) \geq \frac{x^2}{4}. \qquad 2$$

1) a) Under the given assumptions on f ①, there exists $c \in (a,x)$ ① such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ ② ②

b) We first determine the derivatives of f:

$$f'(x) = \pi \cos(\pi x)$$

$$f''(x) = -\pi^2 \sin(\pi x)$$ ③

$$f'''(x) = -\pi^3 \cos(\pi x)$$

Hence, the Taylor polynomial of degree 3 is given by

$$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= 0 - \pi(x-1) + \frac{\pi^3}{3!}(x-1)^3$$

$$= -\pi(x-1) + \frac{\pi^3}{6}(x-1)^3$$ ③

Hence,

$$\sin(3) \approx p_3\left(\frac{3}{\pi}\right) = -\pi\left(\frac{3}{\pi}-1\right) + \frac{\pi^3}{6}\left(\frac{3}{\pi}-1\right)^3$$

$$= \pi - 3 - \frac{(\pi-3)^3}{6}$$ (full marks here, decimal expansion not required) 2

$$\approx 0.1411$$

c) We estimate the error term from Taylor's Theorem.

Since $f^{IV}(x) = \pi^4 \sin(\pi x)$, 1

$|f^{IV}(x)| \leq \pi^4$, so 2

$$\left|\frac{f^{IV}(x)}{4!}\left(\frac{3}{\pi}-1\right)^4\right| \leq \frac{(\pi-3)^4}{24} \approx 1.675 \cdot 10^{-5}$$ 3

Alternatively, note that $c \in \left(\frac{3}{\pi}, 1\right)$, so

$|\sin(\pi c)| \leq \sin 3$ and use that

in the estimate.

(Note that this means you're using your result to get the error bound, so the first, less accurate estimate is cleaner.)

3) Let $m_i = \inf\{ f(x) : x \in [\xi_{i-1}, \xi_i]\}$ 1

and $M_i = \sup\{ f(x) : x \in (\xi_{i-1}, \xi_i]\}$, $i = 1, \dots, n$, 1

(all exist, as f is bounded). 1

Then $$\underline{L}(f, P) = \sum_{i=1}^{n} m_i (\xi_i - \xi_{i-1})$$ 2

and $$U(f, P) = \sum_{i=1}^{n} M_i (\xi_i - \xi_{i-1})$$ 2

a) As f is decreasing,

$$m_i = f(\xi_i) = \frac{1}{1 + i/n} = \frac{n}{n+i}.$$ 2

Also, $\xi_i - \xi_{i-1} = \frac{1}{n}$ for $i = 1, \dots, n$. 1

Hence,

$$L(f, P_n) = \sum_{i=1}^{n} \frac{n}{n+i} \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n+i}.$$ 3

b) f is continuous and hence Riemann-integrable on $[1,2]$. 2 As $\|P_n\| = \frac{1}{n} \to 0$ as 1 $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim_{n\to\infty} L(f, P_n)$$

$$\overset{2}{=} \int_1^2 f(x)\,dx = \ln(2) - \ln(1) = \ln 2 .$$ 2

4a) Let P_n be the given partition and define m_i and M_i as in question 3.

First version: $M_i = m_i$ for all i, except the subinterval(s) containing $\frac{1}{2}$ of which there are at most 2. (4)

Hence, $U(\chi_A, P_n) - L(\chi_A, P_n) = \sum_{i=1}^{n} (M_i - m_i) \frac{1}{n} \leq \frac{2}{n}$ (2)

[only deduct 1 mark if estimate $\frac{1}{n}$ is given]

Second version: Let n be odd. Then $M_i = \begin{cases} 1 & \text{if } i \leq \frac{n+1}{2} \\ 0 & \text{otherwise} \end{cases}$

and $m_i = \begin{cases} 1 & \text{if } i \leq \frac{n-1}{2} \\ 0 & \text{otherwise} \end{cases}$. (4)

Hence, $U(\chi_A, P_n) = \sum_{i=1}^{(n+1)/2} \frac{1}{n} = \frac{n+1}{2} \cdot \frac{1}{n} = \frac{1}{2} + \frac{1}{2n}$

and $L(\chi_A, P_n) = \sum_{i=1}^{(n-1)/2} \frac{1}{n} = \frac{n-1}{2} \cdot \frac{1}{n} = \frac{1}{2} - \frac{1}{2n}$ (2)

Third version: Let n be even. Then $M_i = \begin{cases} 1 & \text{if } i \leq \frac{n}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$,

and $m_i = \begin{cases} 1 & \text{if } i \leq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$. (4)

Hence, $U(\chi_A, P_n) = \sum_{i=1}^{\frac{n}{2}+1} \frac{1}{n} = \left(\frac{n}{2}+1\right) \cdot \frac{1}{n} = \frac{1}{2} + \frac{1}{n}$

$L(\chi_A, P_n) = \sum_{i=1}^{n/2-1} \frac{1}{n} = \frac{n}{2} \cdot \frac{1}{n} = \frac{1}{2}$ (2)

[Various combinations of the versions are possible.]

Given $\varepsilon > 0$, choose $n > 2/\varepsilon$ (or $1/\varepsilon$ in second or third version).

Then

$$\mathcal{U}(\chi_A, P_n) - \mathcal{L}(\chi_A, P_n) \leq \frac{2}{n} < \varepsilon \qquad (2)$$

$$\left(\text{or } \leq \frac{1}{n} < \varepsilon\right)$$

and χ_A is integrable by the Riemann criterion (1)

Moreover, (first version) $m_i = \begin{cases} 1 & \text{for } i \leq n/2 \\ 0 & \text{otherwise} \end{cases}$,

so $\mathcal{L}(\chi_A, P_n) = \sum_{i=1}^{[n/2]} \frac{1}{n} = \begin{cases} 1/2 - 1/2n, & n \text{ odd} \\ 1/2, & n \text{ even} \end{cases}$

(only necessary for first version, as already calculated in other versions).

As χ_A is integrable and $\|P_n\| \to 0$, we have

$$\int_0^1 \chi_A \, dx = \lim_{n \to \infty} \mathcal{L}(\chi_A, P_n) = \frac{1}{2} \qquad (4)$$

[In versions 2 & 3 it is sufficient to argue with odd or even n. Similar arguments involving the upper sums are also possible.]

b) Again use the partitions P_n.

We have $m_i = 0$ for all $i = 1, \dots, n$

and $M_i = \begin{cases} 1 & \text{for the at most 8 subintervals containing } 0, 1/2, 1/2 \text{ and } 2/3 \\ 0 & \text{otherwise} \end{cases}$ (3)

(as each point can be contained in at most two subintervals).

[One could be more specific here, e.g. 0 is always only contained in the first subinterval, but this makes no difference to the argument]

Hence, $L(\chi_A, P_n) = 0$ and $U(\chi_A, P_n) \leq 8 \cdot \frac{1}{n}$. (2)

Given $\varepsilon > 0$, choose $n > 8/\varepsilon$. Then

$U(\chi_A, P_n) - L(\chi_A, P_n) \leq 8 \cdot \frac{1}{n} < \varepsilon$, so

χ_A is integrable by the Riemann criterion. (3)

Moreover, $\int_0^1 \chi_A(x)\, dx = \lim_{n \to \infty} L(\chi_A, P_n) = 0$. (2)

4c) Let P be any partition of $[0,1]$.
As every subinterval contains both rational and irrational points, $M_i = 1$ and $m_i = 0$ for all subintervals. 3 Thus

$L(\chi_A, P) = 0$ and $U(\chi_A, P) = 1$. 2

As P was arbitrary, χ_A is not Riemann-integrable. 2 [Deduct 2 marks if only partitions P_n are used.]

d) We again use the partitions P_n.
Clearly, $L(\chi_A, P_n) = 0$ 2 as $m_i = 0$ for all i.
The idea is now the following:
We want to show that $U(\chi_A, P_n)$ gets arbitrarily small. As the points $\{\frac{1}{k}\}_{k\in\mathbb{N}}$ accumulate near 0, we will have an interval there, where $M_i = 1$. We then need to make sure that there are not too many

other subintervals with $M_i = 1$.

There are again many ways of doing this, but maybe the easiest is the following:

Given $\varepsilon > 0$, choose $n > 3/\varepsilon$. Consider the partition P_{n^2}.

Clearly, for $[0, \frac{1}{n^2}]$ we have $M_1 = 1$. For the next intervals $[\frac{1}{n^2}, \frac{2}{n^2}], \dots, [\frac{n-1}{n^2}, \frac{n}{n^2}]$ we also assume the worst case: $M_i = 1$. Hence we have n subintervals of length $\frac{1}{n^2}$ which we must assume each contain a point of A.

For the remaining subintervals in $[\frac{1}{n}, 1]$ there are n points of A left, which in the worst case each lie on the boundary of two subintervals. Hence, we get at most another $2n$ subintervals of length $\frac{1}{n^2}$ contributing to the upper sum, giving at most $3n$ such subintervals in total.

4d) Therefore,

$$U(\chi_A, P_{n^2}) = \sum_{i=1}^{n^2} M_i \frac{1}{n^2} \leq \frac{3n}{n^2} = \frac{3}{n} < \varepsilon$$ 15

By the Riemann criterion χ_A is integrable 1

and $\int_0^1 \chi_A(x)dx = \lim_{n\to\infty} L(f, P_n) = 0$. 2

Alternatively, (without using P_n) one could argue in the following way to show that $U(f,P)$ can be made arbitrarily small:

Given $\varepsilon > 0$, let $t_1 = \varepsilon/2$. Then $M_1 = \sup\{\chi_A(x) : x \in [0, \varepsilon/2]\} = 1$. Outside $[0, \varepsilon/2]$ there are only finitely many points in A (in fact this will be approximately $2/\varepsilon$). Let N be the number of these points. Partition the interval by choosing intervals of length $\varepsilon/2N$ around each of these N points (not worrying about possible intersections). Then w.r.t. this partition P, we have

$$U(\chi_A, P) \leq \frac{\varepsilon}{2} + N \cdot \frac{\varepsilon}{2N} \leq \varepsilon.$$ 15