Mathematical Analysis (MA552)

Assignment 3

This assignment counts for 5% of the final mark. Full marks will be awarded for 100 marks. You may use any theorems proved in lectures or in the exercise sheets, but you must tell me which theorems you are using, and why the hypotheses they require hold.

DUE DATE Noon, Friday 14th May, 2010

1. (a) State without proof the Mean Value Theorem, specifying all conditions required of the function concerned to guarantee the conclusion of the theorem. [5 marks]

   (b) A function $f : [0, \infty) \to \mathbb{R}$ satisfies the following conditions:

   (I) f is continuous and $f(0) = 0$,

   (II) f is differentiable, $f'(x) > 0$ and $f(x) \leq f'(x)^2$ for all $x > 0$.

   Show that this implies the following:

   i. $f(x) > 0$ for all $x > 0$. [6 marks]

   ii. By considering $g(x) = \sqrt{f(x)}$, show that $f(x) \geq \frac{1}{4}x^2$. [9 marks]

2. Let f be a real valued function of a real variable x. Suppose that f and its first n derivatives are continuous on $[a, x]$ and differentiable on (a, x).

   (a) State without proof Taylor's Theorem giving a polynomial approximation of degree n for $f(x)$. [6 marks]

   (b) Use the Taylor polynomial of degree 3 at $a = 1$ of the function $f(x) = \sin(\pi x)$ to give an approximation to $\sin(3)$. [8 marks]

   (c) Show that the error in (b) is less than $2 \cdot 10^{-5}$. (You are expected to use the error term from Taylor's Theorem here.) [6 marks]

3. Given a bounded function $f : [a, b] \to \mathbb{R}$, and a partition

$$\mathcal{P} : a = \xi_0 < \xi_1 < \xi_2 < ... < \xi_n = b,$$

   define the *lower Riemann sum* $L(f, \mathcal{P})$ and the *upper Riemann sum* $U(f, \mathcal{P})$ of f with respect to $\mathcal{P}$. [7 marks]

   (a) Determine the lower Riemann sum for $f(x) = 1/x$ on $[1, 2]$ with respect to the partition $\mathcal{P}_n$ given by $1 = \xi_0 < \xi_1 < \xi_2 < ... < \xi_n = 2$ with $\xi_i = 1 + \frac{i}{n}$ for $i = 0, ..., n$ and $n \in \mathbb{N}$. [6 marks]

(b) Use this result to show

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \ln 2.$$

[7 marks]

4. For a set $A \subseteq \mathbb{R}$ we define its characteristic function by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \quad \text{otherwise.} \end{cases}$$

For the following sets A, determine whether χ_A is Riemann integrable over $[0,1]$ and calculate $\int_0^1 \chi_A(x)dx$ when possible.

(a) $A = [0, \frac{1}{2}]$. [13 marks]

(b) $A = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. [10 marks]

(c) $A = [0,1] \setminus \mathbb{Q}$. [7 marks]

*(d) $A = \{\frac{1}{k} : k \in \mathbb{N}\}$. [20 marks]

Hint: In most cases, it is simplest to consider partitions

$$\mathcal{P}_n : 0 < \frac{1}{n} < \frac{2}{n} < ... < \frac{n}{n-1} < 1,$$

and use the Riemann criterion (Theorem 17).