

# Mathematical Analysis (MA552)

## Assignment 2

This assignment counts for 5% of the final mark. Attempt all questions. You may use any theorems proved in lectures or in the exercise sheets, but you must tell me which theorems you are using, and why the hypotheses they require hold.

**DUE DATE Noon, Friday 19th March, 2010**

1. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$g(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 0, \\ x^3 - 2x + 1 & \text{if } x > 0. \end{cases}$$

- (a) Use the  $(\varepsilon, \delta)$ -definition of continuity to show that  $g$  is continuous at  $x = 1$ . [7 marks]
- (b) Show that  $g$  is discontinuous at  $x = 0$ . [5 marks]

2. Using only that  $1$ ,  $x$  and  $\sin(x)$  are continuous on  $[0, 1]$  and differentiable on  $(0, 1)$  with derivatives  $0$ ,  $1$  and  $\cos(x)$ , respectively, show that there is a **unique**  $c \in (0, 1)$  such that

$$\sin^2(c) = \frac{1}{1 + c^2}.$$

[16 marks]

3. Let

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- (a) Using the definition of the derivative show that  $f$  is differentiable at  $0$  and determine  $f'(0)$ . [5 marks]
- (b) Is  $g$  differentiable at  $0$ ? Justify your answer. [5 marks]
- (c) Show that  $f'(x)$  and  $g'(x)$  exist for  $x \neq 0$  and determine their values. [6 marks]

4. Let  $f$  be continuous on  $[0, 1]$  and such that  $f(x) \neq 0$  for all  $x \in [0, 1]$ . Prove that  $1/f$  is continuous and bounded. [6 marks]