Solutions
& marking scheme

MA552 Analysis – Class Test

NAME:

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Attempt all FOUR questions.

General notes
- give benefit of doubt
- err on the generous side

MUST USE DEF'N

Q1. Prove from the definition of convergence that

$$\frac{3n+(-1)^n}{12n+21} \to \frac{1}{4}$$

[5 marks]

Let $\varepsilon > 0$ be given. 1

Setting $N(\varepsilon) > \frac{1}{\varepsilon}$. 1 any correct

we have for all $n > N$ $\frac{1}{2}$ that

$$\left| \frac{3n+(-1)^n}{12n+21} - \frac{1}{4} \right| = \left| \frac{4(-1)^n - 21}{4(12n+21)} \right| \quad \frac{1}{2}$$

$$\leq \frac{25}{4(12n+21)}$$ by Triangle Inequality 1

$$< \frac{25}{36n}$$

$$< \frac{1}{n}$$

$$< \frac{1}{N}$$

1 any correct sequence of inequalities

$< \varepsilon$ as required

Note If use Algebra of Limits, give zero.

Q2. Find the limits of the following sequences. You may use any theorem proved in lectures or in the exercise sheets, but you must state the theorem being used and show that any hypotheses of the theorems used hold.

(i)

$$x_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n}$$

(ii)

$$x_n = \frac{(4n+1)^{25}}{\sqrt{3n}} \frac{\sqrt{7n+3}}{(5n+2)^{25}}$$

[5 marks]

(i) $$x_n = \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} \cdot \sqrt{n}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

1 There are other possibilities but must show x_n in a form to which Alg of Limits Thm applies.

So $$\lim x_n = \frac{1}{\sqrt{1+\lim\frac{1}{n}}+1} = \frac{1}{2}$$

$\frac{1}{2}$ for the correct limit

1 (by the Algebra of Limits Theorem, since the limits exist and the denominator $\neq 0$ including in the limit, and since if

1 ($y_n \geq 0$, $y_n \to l$ then $\sqrt{y_n} \to \sqrt{l}$.

any correct statement of the Theorems.

(Space for Q2 continued)

(ii) $$x_n = \left(\frac{4n+1}{5n+2}\right)^{25} \sqrt{\frac{7n+3}{3n}}$$

$$= \left(\frac{4+\frac{1}{n}}{5+\frac{2}{n}}\right)^{25} \sqrt{\frac{7+\frac{3}{n}}{3}} \qquad 1$$

so $$\lim x_n = \left(\frac{4+0}{5+0}\right)^{25} \sqrt{\frac{7+0}{3}}$$

$$= \left(\frac{4}{5}\right)^{25} \sqrt{\frac{7}{3}} \qquad \frac{1}{2}$$

- If statement of any theorems not in (i) ~~the~~ but here, give the marks.

Q3. Consider the sequence

$$x_n = \frac{4^n + (-1)^n n!}{7^n + 5(n!)}$$

Decide whether the sequence converges or not. If it converges, give a proof. If it does not converge, say why it does not.

$$x_n = \frac{\frac{4^n}{n!} + (-1)^n}{\frac{7^n}{n!} + 5} \quad (1)$$

Since $\underline{\frac{x^n}{n!} \to 0 \text{ all } x \in \mathbb{R},}$ we (1) have that

$$X_{2n} \to \frac{1}{5}$$

$$X_{2n+1} \to -\frac{1}{5}$$

might say oscillates or $X_n \sim (-1)^n \frac{1}{5}$ which is OK for partial marks if needed

(3) for any statement that conveys the idea of needing a unique limit

Sequence does NOT converge, as Every subsequence of a convergent sequence has the same limit

OR a convergent sequence has a unique limit

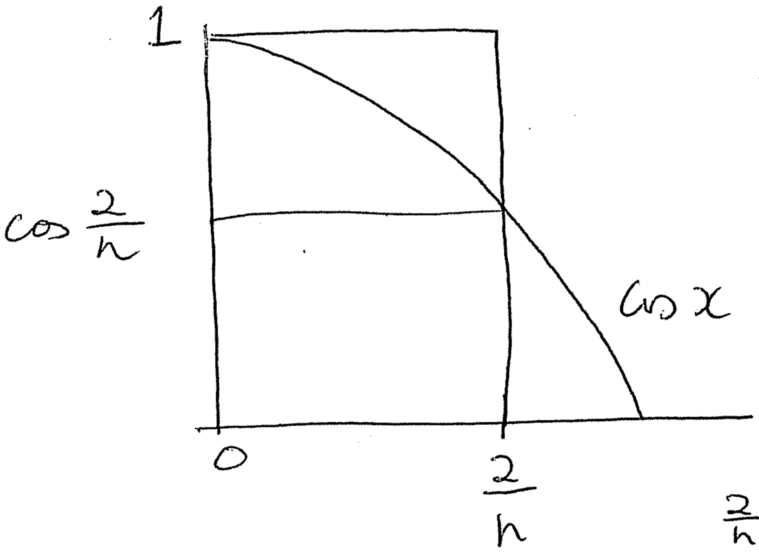
OR...

Q4. Draw the graph of $\cos x$ on the interval $[0, \frac{2}{n}]$. By comparing the area under the curve with both the upper and lower areas, show that

$$n \sin \frac{2}{n} \to 2,$$

justifying your reasoning. [5 marks]

MUST USE THIS METHOD



(2) for correct & labelled diagram

$$\text{lower area} < \int_0^{\frac{2}{n}} \cos x \, dx < \text{upper area}$$

give partial marks if needed

i.e. $$\frac{2}{n} \cos\left(\frac{2}{n}\right) < \sin\left(\frac{2}{n}\right) < \frac{2}{n}$$

so $$\underbrace{2 \cos\left(\frac{2}{n}\right)}_{\to 2} < n \sin\left(\frac{2}{n}\right) < 2$$ (2)

as $\cos(0) = 1$

By the <u>Sandwich Theorem,</u> (1)

$$n \sin\left(\frac{2}{n}\right) \to 2$$

<u>Note</u> Equiv to show $\frac{n}{2} \sin\left(\frac{2}{n}\right) \to 1$ . . . . . .