Mathematical Analysis (MA552)

Assignment 2

This assignment counts for 5% of the final mark. Attempt all questions. You may use any theorems proved in lectures or in the exercise sheets, but you must tell me which theorems you are using, and why the hypotheses they require hold.

DUE DATE Noon, Friday 19th March, 2010

1. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by
$$g(x) = \begin{cases} x^2 - 3 & \text{if } x \leq 0, \\ x^3 - 2x + 1 & \text{if } x > 0. \end{cases}$$

   (a) Use the (ε, δ)-definition of continuity to show that g is continuous at $x = 1$. [7 marks]

   (b) Show that g is discontinuous at $x = 0$. [5 marks]

2. Using only that 1, x and $\sin(x)$ are continuous on $[0, 1]$ and differentiable on $(0, 1)$ with derivatives 0, 1 and $\cos(x)$, respectively, show that there is a **unique** $c \in (0, 1)$ such that
$$\sin^2(c) = \frac{1}{1 + c^2}.$$
[16 marks]

3. Let
$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
and
$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Using the definition of the derivative show that f is differentiable at 0 and determine $f'(0)$. [5 marks]

   (b) Is g differentiable at 0? Justify your answer. [5 marks]

   (c) Show that $f'(x)$ and $g'(x)$ exist for $x \neq 0$ and determine their values. [6 marks]

4. Let f be continuous on $[0, 1]$ and such that $f(x) \neq 0$ for all $x \in [0, 1]$. Prove that $1/f$ is continuous and bounded. [6 marks]

1. (a) Let $\varepsilon > 0$ be given and choose $\delta = \min\{1, \varepsilon/5\}$ 3

Then for $|x-1| < \delta$,

$|g(x) - g(1)| = |x^3 - 2x + 1 - 0| = |x^3 - 2x + 1|$. 1

By long division,

$x^3 - 2x + 1 = (x-1)(x^2 + x - 1)$. 1

and for $|x-1| < 1$, $|x^2 + x - 1| \leq 5$. 1

Hence,

$|g(x) - g(1)| = |x-1|\ |x^2+x-1| < 5\delta < \varepsilon$. 1

(Various other choices of δ are also possible).

(b) We consider the right and left limits of g at 0:

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x^3 - 2x + 1) = 1$, 2

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x^2 - 3) = -3$

As the two limits do not coincide, g is 3 not continuous at 0 (by the limit characterisation of continuity).

(Rather than using the left limit, arguing that $g(0) = -3 \neq \lim_{x \to 0^+} g(x)$ also shows discontinuity of g by the same result as above)

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By long division,

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Hence,

$|g(x) - g(1)| = |x-1|\,|x^2+x-1| < 5\delta < \varepsilon$. (1)

(Various other choices of δ are also possible).

(b) We consider the right and left limits of g at 0:

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x^3 - 2x + 1) = 1$, (2)

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (x^2 - 3) = -3$

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(Rather than using the left limit, arguing that $g(0) = -3 \neq \lim_{x \to 0^+} g(x)$ also shows discontinuity of g by the same result as above)

2.) • As $\sin(x)$ is continuous and products of continuous functions are continuous, $\sin^2(x)$ is continuous. 1

• By the same result, x^2 is continuous. 1

• As sums of continuous functions are continuous 1
$1+x^2$ is continuous.

• Since $1+x^2 \neq 0$, $\frac{1}{1+x^2}$ is continuous. 1

• Again using continuity of sums and differences of continuous functions, we get that

$$f(x) = \sin^2(x) - \frac{1}{1+x^2}$$

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is continuous on $[0,1]$.

By the same arguments for differentiability, f is differentiable on $(0,1)$. 1

1

We have $f(0) = -1$ 1 and $f(1) = \sin^2 1 - \frac{1}{2} \approx 0.708 > 0$.

By the Intermediate Value Theorem (IVT), there exists $c \in (0,1)$ such that $f(c) = 0$, ie

$$\sin^2 c = \frac{1}{1+c^2}.$$

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$$\sin^2 c = \frac{1}{1+c^2}.$$ 2

3a) $\frac{f(h)-f(0)}{h} = \frac{h^3 \sin 1/h}{h} = h^2 \sin \frac{1}{h}$ (2)

and $|h^2 \sin 1/h| \le |h|^2 \longrightarrow 0$ as $h \to 0$ (2)

Hence, $\lim_{h \to 0} \frac{f(h)-f(0)}{h} = 0$ and $f'(0) = 0$. (1)

b) $\frac{g(h)-g(0)}{h} = \frac{h \sin 1/h}{h} = \sin 1/h$. (2)

Let $h_n = \frac{1}{n\pi}$ and $\tilde{h}_n = \frac{1}{(n+1/2)\pi}$ for $n \in \mathbb{N}$.

Then $h_n \to 0$ and $\tilde{h}_n \to 0$ as $n \to \infty$,

but $\sin 1/h_n = \sin(n\pi) = 0$, while

$\sin 1/\tilde{h}_n = \sin(n+1/2)\pi$ oscillates between ± 1.

Hence $\lim_{h \to 0} \frac{g(h)-g(0)}{h}$ does not exist (2) and

g is not differentiable at 0. (1)

(It would actually suffice to just consider the sequence $(\tilde{h}_n)$, as $\sin 1/\tilde{h}_n$ does not converge.)

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b) $\frac{g(h)-g(0)}{h} = \frac{h \sin 1/h}{h} = \sin 1/h$. 2

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4. As f is continuous on $[0,1]$ and $f(x) \neq 0$ for all $x \in [0,1]$, $1/f$ is continuous on $[0,1]$. 2

As a continuous function on a closed bounded interval, $1/f$ is bounded by the Boundedness Theorem. 4