MA552 Analysis Assignment 1

This assignment counts for 5% of the final mark. Attempt all THREE questions. You may use any theorems proved in lectures or in the exercise sheets, but you must tell me which theorems you are using, and why the hypotheses they require hold. Question 1 is worth 7 marks, Question 2 is worth 5 and Question 3 is worth 8.

DUE DATE Noon, 29rd JANUARY, 2010

Q1. State whether the following series converge or diverge. Justify your reasoning.

$$\text{(i)} \quad \sum (-1)^n \frac{2\sqrt{n}}{3n^2+5} \qquad \text{(ii)} \quad \sum \frac{2n}{3n+4}$$

$$\text{(iii)} \quad \sum \frac{2^n n^n}{3(n!)+4}$$

Q2. By comparing

$$\sum_{m=2}^{N} \frac{1}{m(\log m)^2}$$

with the integral

$$\int_2^N \frac{dx}{x(\log x)^2}$$

determine whether the series converges or diverges. Justify your reasoning.

Q3. Suppose Σa_n and Σb_n are infinite series with positive terms, and that

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$$

for each n. Prove that

$$a_n \geq \frac{a_1}{b_1} b_n$$

for each n. Deduce that Σa_n diverges if Σb_n diverges.

When $a_n = \left\{ \dfrac{1.3.5\ldots(2n-1)}{2^n n!} \right\}^2$ show that

1. $\dfrac{a_{n+1}}{a_n} = \left(\dfrac{2n+1}{2n+2} \right)^2$
2. the ratio test can not be used to decide whether Σa_n converges or diverges
3. $\dfrac{a_{n+1}}{a_n} \geq \dfrac{n}{n+1}$ for each n
4. Σa_n diverges.

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MA552 Assignment 1 2009/10

Solutions & Marking Scheme

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Q1 (i) Solution 1

2 $\quad 0 < |a_n| = \dfrac{2\sqrt{n}}{3n^2+5} < \dfrac{2}{3n^{3/2}}$ $\quad$ 1/2

and $\sum \dfrac{1}{n^{3/2}}$ converges 1/2 Hence series converges absolutely 1/2 with comparison 1/2 to $\sum \dfrac{1}{n^{3/2}}$ & hence converges.

Alternate Solution 2

Set $\quad a_n = \dfrac{2\sqrt{n}}{3n^2+5}$

Then (i) $a_n > 0$ $\quad$ 1/2

(ii) $a_n \longrightarrow 0$ $\quad$ 1/2

(iii) $a_{n+1} < a_n$ $\quad$ 1/2

as $\dfrac{d}{dx}\dfrac{2\sqrt{x}}{3x^2+5}$

$$= -\frac{9x^2-5}{2\sqrt{x}(3x^2+5)^2} < 0$$

for $x \geq 1$

So $\sum a_n$ converges by alternating test. $\frac{1}{2}$

Q1 (ii) $a_n = \dfrac{2n}{3n+4} \longrightarrow \dfrac{2}{3} \neq 0$ 1

$\boxed{1\tfrac{1}{2}}$ & hence series diverges. $\tfrac{1}{2}$

(iii) $a_n = \dfrac{2^n n^n}{3n!+4}$

$\boxed{3\tfrac{1}{2}}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)^{n+1}}{3(n+1)!+4} \cdot \frac{3n!+4}{2^n n^n}$$

$$= 2 \cdot \left(\frac{n+1}{n}\right)^n (n+1) \cdot \frac{3+4/n!}{3(n+1)+4/n!}$$

$$\rightarrow \cancel{2/e}$$

$$= 2\left(\frac{n+1}{n}\right)^n \cdot \frac{3+4/(n\cancel{+1})!}{3+4/(n+1)!}$$

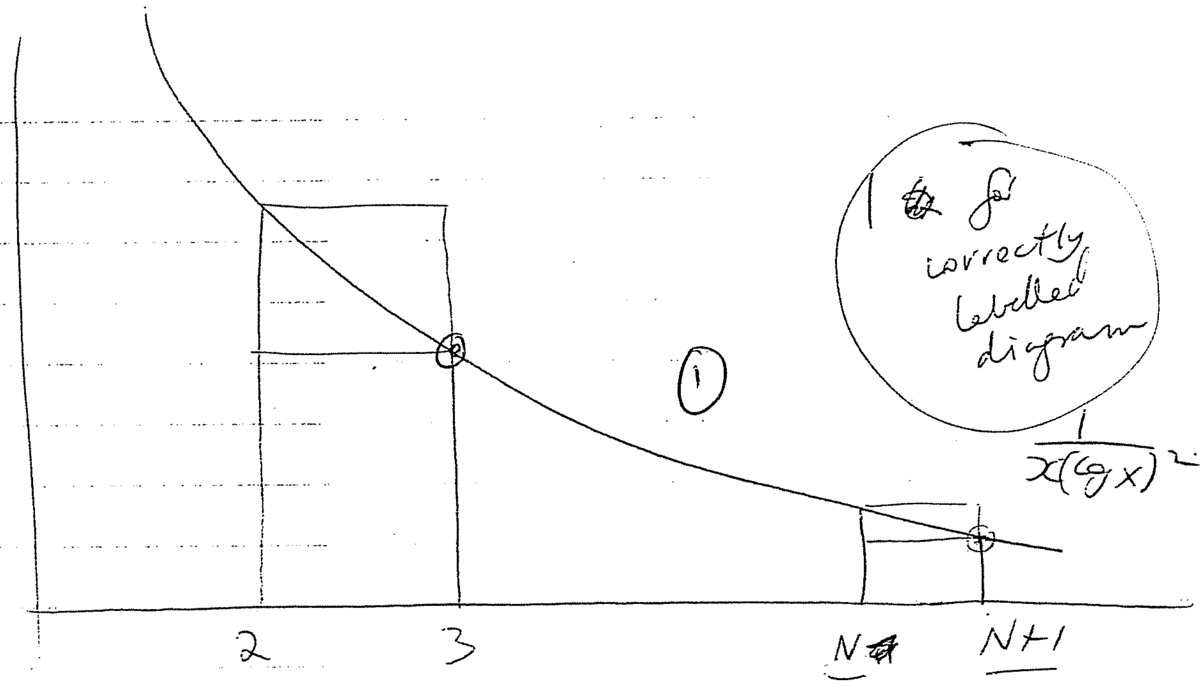
$$\rightarrow 2 \cdot e$$ 2

$$> 1$$

So diverges $\tfrac{1}{2}$ by the ratio test. 1

Q2

(a)



$$\frac{d}{dx}\, x(\log x)^2 = (\log x)^2 + 2\log x \quad > 0$$

$\frac{2\log x}{x} + \frac{2}{x} > 0$

so fn is <u>decreasing</u>.

(1) 1

$$\text{lower sum} \quad < \int_2^{N+1} \frac{1}{x(\log x)^2}\,dx \quad < \text{upper sum}$$

(or its equivalent)

$$\frac{1}{3(\log 3)^2} + \dots + \frac{1}{(N+1)(\log(N+1))^2} < -\frac{1}{\log x}\Big]_2^{N+1}$$

$$\sum_{m=3}^{N+1} \frac{1}{m(\log m)^2} + \frac{1}{2(\log 2)^2} \quad = \frac{1}{\log 2} - \frac{1}{\log(N+1)}$$

$$< \frac{1}{\log 2} + \frac{1}{2(\log 2)^2}$$

(1) 1 for correct inequality

Hence $\boxed{\sigma_N}$ is increasing series bounded above by $\frac{1}{\log 2} + a_2$ & hence the series converges.

(2) 2 for explicitly stating the logic.

Q3) We show $a_n \geq \frac{a_1}{b_1} b_n$ by

math'l induction $\frac{1}{2}$

Step 1 $a_1 \geq \frac{a_1}{b_1} \cdot b_1$ is true $\frac{1}{2}$

(or a_2, b_2 case)

Step 2 assume $a_k \geq \frac{a_1}{b_1} b_k$ $\frac{1}{2}$

Step 3 Have $\frac{a_{k+1}}{a_k} \geq \frac{b_{k+1}}{b_k}$

$$\therefore \quad a_{k+1} \geq a_k \cdot \frac{b_{k+1}}{b_k}$$

$$\geq \frac{a_1}{b_1} \cdot b_k \cdot \frac{b_{k+1}}{b_k} \qquad \boxed{\text{inductive step}}$$

$$= \frac{a_1}{b_1} b_{k+1} \qquad 1$$

alternatively, 2 marks for

$a_n \geq a_{n-1} \cdot \frac{b_n}{b_{n-1}}$

$\geq a_{n-2} \cdot \frac{b_{n-1}}{b_{n-2}} \cdot \frac{b_n}{b_{n-1}}$

$\geq \cdots \geq \frac{a_1}{b_1} \cdot b_n$

as required.

$$\left[\text{Hence } \sum_1^N a_n \geq \sum_1^N \frac{a_1}{b_1} b_n = \frac{a_1}{b_1} \sum_1^N b_n \right]$$

Hence if $\sum b_n$ diverges, so does $\sum a_n$ by the comparison test. 1

Let $a_n = \left(\dfrac{1.3.5. \ldots (2n-1)}{2^n \, n!} \right)^2$

(1)

Then $\dfrac{a_{n+1}}{a_n} = \left(\dfrac{1.3.5 \ldots (2n-1)(2n+1)}{2^{n+1} \, (n+1)!} \cdot \dfrac{2^n \, n!}{1.3.5 \ldots (2n-1)} \right)^2$

$$= \left(\frac{2n+1}{2(n+1)} \right)^2 \qquad 1$$

as required.

(2) $\lim \dfrac{a_{n+1}}{a_n} = 1^2 = 1$

so no conclusion from the ratio test. 1

(3) $\dfrac{a_{n+1}}{a_n} = \left(1 - \dfrac{1}{2(n+1)} \right)^2$

$$= 1 - \frac{1}{n+1} + \frac{1}{4(n+1)^2}$$

$$> 1 - \frac{1}{n+1} \qquad 1$$

$$= \frac{n}{n+1}$$

4 Set $b_n = \frac{1}{n}$ 1 then $\dfrac{a_{n+1}}{a_n} > \dfrac{n}{n+1} = \dfrac{b_{n+1}}{b_n}$

Hence $\sum a_n$ diverges since $\sum \frac{1}{n}$ diverges. $\frac{1}{2}$