An $\mathrm{SL}_2(k)$ Character Variety Whose Dimension Jumps in Characteristic 2

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1 Introduction

In this note we give an example of a finitely generated group Γ such that:
(I) there exists a field Ω of characteristic 2 with a discrete valuation ν_0 and a representation $\rho_0 \colon \Gamma \to \mathrm{SL}_2(\Omega)$ such that the induced action of Γ on the Bruhat-Tits tree of $\mathrm{SL}_2(\Omega)$ tree is fixed-point free;
(II) for every field K of characteristic zero with a discrete valuation, no homomorphism $\rho \colon \Gamma \to \mathrm{SL}_2(K)$ gives rise to a fixed-point free action of Γ on the Bruhat-Tits tree of $\mathrm{SL}_2(K)$. In particular, Γ admits no ANI-representations.
For terminology and background discussion, see [SZ].

If Γ were the fundamental group of a compact 3-manifold M then ρ_0 would give an essential surface in M that is not detectable by the method of ANI representations or by looking at ideal points of curves in the character variety $\mathrm{C}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$. I do not know whether such a 3-manifold exists for this particular choice of Γ, but perhaps a similar example can be constructed for appropriate M.

2 Character Varieties

Let F be a finitely generated group and let k be an algebraically closed field. We consider representations (i.e. group homomorphisms) of F into $\mathrm{SL}_2(k)$. Below, $T \subset \mathrm{SL}_2(k)$ denotes the group of diagonal matrices.

The set $\mathrm{R}(F, \mathrm{SL}_2(k))$ of representations (i.e. group homomorphisms) $\rho \colon F \to \mathrm{SL}_2(k)$ has the structure of an affine variety over k. The group $\mathrm{SL}_2(k)$ acts on

$\mathrm{R}(F, \mathrm{SL}_2(k))$ by conjugation: $(D.\rho)(\gamma) := D\rho(\gamma)D^{-1}$. Since $\mathrm{SL}_2(k)$ is a reductive algebraic group, we have a well-defined quotient variety $\mathrm{C}(F, \mathrm{SL}_2(k))$, the points of which correspond to Zariski-closed conjugacy classes of representations. By results of R. W. Richardson, the conjugacy class of a representation ρ is closed if and only if ρ is completely reducible. Clearly ρ is completely reducible if and only if either $\rho(F)$ lies in some conjugate of T or ρ is irreducible. The co-ordinate ring of the affine variety $\mathrm{C}(F, \mathrm{SL}_2(k))$ is the ring of conjugation-invariant regular functions on $\mathrm{R}(F, \mathrm{SL}_2(k))$. One way to obtain such a function is as follows: given $\gamma \in \Gamma$, define $\mathrm{Tr}_\gamma \colon \mathrm{R}(\Gamma, \mathrm{SL}_2(k)) \to k$ by $\mathrm{Tr}_\gamma(\rho) = \mathrm{Tr}\ \rho(\gamma) = \chi_\rho(\gamma)$, where Tr denotes trace and $\chi_\rho \colon \Gamma \to k$ is the character of ρ. We denote by π the canonical map from $\mathrm{R}(F, \mathrm{SL}_2(k))$ to $\mathrm{C}(F, \mathrm{SL}_2(k))$.

The rest of this section is not essential for Section 3. Let $\mathcal{S}$ be the k-algebra generated by functions of the form Tr_γ, a subring of the co-ordinate ring of $\mathrm{C}(\Gamma, \mathrm{SL}_2(k))$. If $\mathcal{S}$ is finitely generated as a k-algebra then $\mathcal{S}$ is the co-ordinate ring of an affine variety $\mathrm{X}(F, \mathrm{SL}_2(k))$, and we have a dominant map $\psi \colon \mathrm{C}(F, \mathrm{SL}_2(k)) \to \mathrm{X}(F, \mathrm{SL}_2(k))$. It is known that if $k = \mathbb{C}$ then $\mathcal{S}$ is indeed finitely generated and ψ is an isomorphism (hence the name "character variety" or "variety of characters" for $\mathrm{C}(F, \mathrm{SL}_2(k))$). For general k it follows from results of Donkin that $\mathcal{S}$ is a finitely generated k-algebra and the map ψ is finite (see [M], Theorem 1.4); note that any regular conjugation-invariant function on $\mathrm{SL}_2(k)$ is a polynomial in Tr.

Conjecture 2.1 *The map ψ is an isomorphism.*

3 The Example

Let Γ be the group with presentation

$$\Gamma = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, [b, cac] = [b, aca] = 1, cac = aca \rangle.$$

Note that a, c generate a copy of the symmetric group S_3. We show that:
(a) if the characteristic of k is not 2 then the variety $\mathrm{C}(\Gamma, \mathrm{SL}_2(k))$ is finite, and for every $\rho \in \mathrm{R}(\Gamma, \mathrm{SL}_2(k))$, $\rho(\Gamma)$ is finite;
(b) if k has characteristic 2 then $\mathrm{X}(\Gamma, \mathrm{SL}_2(k))$ is infinite: in particular, $\mathrm{C}(\Gamma, \mathrm{SL}_2(k))$ is infinite and there exists an affine curve in $\mathrm{C}(\Gamma, \mathrm{SL}_2(k))$ with an ideal point. (Applying the construction described in [SZ] now gives the field Ω, the discrete valuation ν_0 and the representation ρ_0 of (I) above.)

Because a finite group acting without inversions on a tree must fix a vertex ([S], I.6, Example 3.1), (a) implies (II) above.

Assume that the characteristic of k is not 2. Let $\rho\colon \Gamma \to \mathrm{SL}_2(k)$ be any completely reducible representation. Set $A = \rho(a), B = \rho(b), C = \rho(c)$. The hypothesis on the characteristic implies that $A, B, C \in \pm I$, and part (a) follows.

Now assume that k has characteristic 2. For $\mu \in k$, define $\rho_\mu\colon \Gamma \to \mathrm{SL}_2(k)$ by

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho(b) = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \rho(c) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\rho(c)\rho(a)\rho(c) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and

$$\rho(a)\rho(c)\rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so ρ is well-defined. Now $\mathrm{Tr}_{bc}(\rho_\mu) = \mathrm{Tr}\ (\rho_\mu(bc)) = \mu$, so the function Tr_{bc} takes on infinitely many values. It follows that $\mathrm{X}(\Gamma, \mathrm{SL}_2(k))$ is infinite, as required.

References

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