Compactifications of a Representation Variety

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Abstract

Let F be a finitely generated group and let G be a linear algebraic group over an algebraically closed field k. Let $\mathrm{R}(F,G)$ be the variety of representations of F in G. Given a finite set of generators Δ for F, we define a compactification $\mathrm{R}_\Delta(F,\overline{G})$ of $\mathrm{R}(F,G)$. The compactification is highly dependent on the choice of generators. If F' is another finitely generated group with a finite set of generators Δ' and $\phi: F' \to F$ is a homomorphism, then there is an induced morphism of varieties $\phi^\#: \mathrm{R}(F,G) \to \mathrm{R}(F',G)$. We prove that if $\phi(\Delta' \cup \{1\}) \subset \Delta \cup \{1\}$, then $\phi^\#$ extends to a morphism from $\mathrm{R}_\Delta(F,\overline{G})$ to $\mathrm{R}_{\Delta'}(F',\overline{G})$. We study the morphisms arising in this way from a group extension $1 \to N \to F \to Q \to 1$.

1 Introduction

Let F be a finitely generated group and let G be a linear algebraic group. The set of representations (i.e. abstract group homomorphisms) from F to G is an affine variety called the **representation variety** $\mathrm{R}(F,G)$. If G is reductive then G acts on $\mathrm{R}(F,G)$ by conjugation, and the quotient variety $\mathrm{C}(F,G)$, called the **character variety**, is also affine. These varieties have been the subject of much study (see [8], [2], [6], for example). In this paper we describe a natural way to compactify representation varieties.

A homomorphism $\phi: F' \to F$ of finitely generated groups gives rise to a morphism $\phi^\#: \mathrm{R}(F,G) \to \mathrm{R}(F',G)$, defined by $\phi^\#(\rho) = \rho \circ \phi$. If G is reductive then we also have a morphism $\overline{\phi^\#}$ from $\mathrm{C}(F,G)$ to $\mathrm{C}(F',G)$ determined by ϕ. The correspondences $F \mapsto \mathrm{R}(F,G)$, $\phi \mapsto \phi^\#$ and $F \mapsto \mathrm{C}(F,G)$, $\phi \mapsto \overline{\phi^\#}$ are contravariant functors. Maps of the form $\phi^\#$ arise in many settings. In [9], the author showed that the real polarization map of Weitsman [13] may be interpreted using this formalism. Slodowy [12], §II.4 studied the fibres of $i^\#$, where i is the inclusion of a Sylow p-subgroup of a finite group Γ into Γ.

We define a **compactification** of an affine variety V to be a projective variety W that contains V as an open subset. (Usually one requires also that V

2000 *Mathematics Subject Classification*: 20C99 should be dense in W, but it will be convenient for us to adopt this more general definition. If V is dense then we shall call W a **strict compactification.**) If W_1, W_2 are compactifications of V then we define a morphism (isomorphism) of compactifications to be a morphism (isomorphism) of varieties $\psi: W_1 \to W_2$ that restricts to the identity on V. When W_1, W_2 are strict compactifications, there can exist at most one morphism $\psi: W_1 \to W_2$ (since any two must agree on the dense open subset V). If V carries some extra structure, such as the action of some group, then W can often be chosen to be compatible with this structure.

Compactifications of representation varieties and character varieties have been constructed before in a slightly different setting. Instead of $\mathrm{R}(F,G)$ and $\mathrm{C}(F,G)$, one may define real analytic varieties $\mathrm{R}(F,H)$ and $\mathrm{C}(F,H)$ for H a real reductive Lie group (see [2]). Let Π^g be the fundamental group of a compact orientable surface of genus $g \geq 2$. Teichmüller space $\mathcal{T}_g$ may be identified with a subspace of $\mathrm{C}(\Pi^g, \mathrm{PSL}_2(\mathbb{R}))$, and several compactifications of $\mathcal{T}_g$ are known, including the Thurston compactification [5].

Our aim is to construct compactifications of $\mathrm{R}(F,G)$ and $\mathrm{C}(F,G)$ for arbitrary F that are compatible with the maps $\phi^\#$ and $\overline{\phi^\#}$. One motivation for this is as follows. Casson's invariant for an integral homology 3-sphere M is defined to be the topological intersection number of a certain pair of subvarieties V_1 and V_2 of $\mathrm{C}(\Pi^g, \mathrm{SU}(2))$ (see [1] for details). Each V_i is the image of $\overline{p_i^\#}$ for some epimorphism p_i from Π^g to a free group of rank g. A natural variation on this idea would be to consider the algebraic intersection number of the analogous pair of subvarieties in a compactification of $\mathrm{C}(\Pi^g, G)$, where G is a complex reductive algebraic group.

As a first step, in this paper we construct compactifications of the representation variety $\mathrm{R}(F,G)$. Unfortunately, the requirement that $\phi^\#$ extend to a morphism of compactified representation varieties is too strong (see Theorem 5.7); to fulfill it, we are forced to consider not just a single compactification of $\mathrm{R}(F,G)$, but one for each finite set $\Delta = \{\gamma_1, \ldots, \gamma_r\}$ of generators for F. First we choose a suitable compactification $\overline{G}$ of G. The map $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_r))$ gives an embedding of $\mathrm{R}(F,G)$ as a closed subset of G^r, and it is well known that the structure of affine variety that $\mathrm{R}(F,G)$ inherits is independent of the choice of generating set. Given a generating set Δ for F as above, we define $\mathrm{R}_\Delta(F, \overline{G})$ to be roughly the set of functions of the form $\rho: U \to \overline{G}$, for some U with $\Delta \subset U \subset F$, such that ρ is a homomorphism wherever this makes sense. Then $\mathrm{R}_\Delta(F, \overline{G})$ embeds as a closed subvariety of the projective variety $\overline{G}^r$ and $\mathrm{R}(F,G)$ is open in $\mathrm{R}_\Delta(F, \overline{G})$. Moreover, given groups F, F' with finite generating sets Δ, Δ' respectively and a homomorphism $\phi: F' \to F$ such that $\phi(\Delta' \cup \{1\}) \subset \Delta \cup \{1\}$, the morphism $\phi^\#$ extends to a morphism $\Phi^\#$ from $\mathrm{R}_\Delta(F, \overline{G})$ to $\mathrm{R}_{\Delta'}(F', \overline{G})$.

Let

$$1 \to N \to F \xrightarrow{p} Q \to 1$$

be an extension of groups, where F is finitely generated. Then Q is finitely generated and $p^\#: \mathrm{R}(Q,G) \to \mathrm{R}(F,G)$ is a closed embedding with image consisting of the set of representations ρ such that $\rho|_N$ is trivial. We may replace N by a suitable finitely generated subgroup K of N; for by Lemma 6.1, there exists K such that

$$p^{\#}(\mathrm{R}(Q,G)) = (j^{\#})^{-1}(\rho_0^K),$$

where j is the inclusion of K in F and ρ_0^K denotes the trivial representation of K. We call such a subgroup K an **approximate G-kernel** of p.

We show how to carry this idea over to the case of compactified representation varieties. We define the notion of an approximate $\overline{G}$-kernel (Definition 6.4), and we prove that approximate $\overline{G}$-kernels always exist. They are highly dependent on the choice of generating sets for the finitely generated groups concerned.

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2 Compactifying G

We begin with some notation. The symbol F denotes a finitely generated group. The identity of a group is denoted by 1, or by I if the group is a matrix group. We write F_s for the free group on s generators. The subgroup generated by a subset S of a group is denoted by $\langle S\rangle$. All varieties (including algebraic groups) are defined over a fixed algebraically closed field k, and we allow varieties to be reducible. We set $k^* = k\backslash\{0\}$. If V is a vector space over k with $\dim V < \infty$ then $\mathbb{P}(V)$ denotes the corresponding projective space. We denote the space of $n \times n$ matrices with entries from k by $\mathrm{M}_n(k)$.

We fix a linear algebraic group G. For material on representation varieties, see the article [8] of Lubotzky and Magid. Usually in the theory of representation varieties it is assumed that G is connected and reductive, but the basic results and definitions carry over to arbitrary G with simple modifications. However, the quotient variety $\mathrm{C}(F,G)$ is only defined for reductive G.

We choose a strict compactification $\overline{G}$ of G as follows. Pick an embedding α of G as a closed subgroup of $\mathrm{PGL}_n(k)$. Such an embedding always exists, for we can find an embedding $G \to \mathrm{GL}_m(k)$ for some m, and then follow this with the mapping $\mathrm{GL}_m(k) \stackrel{\xi_m}{\to} \mathrm{GL}_{m+1}(k) \to \mathrm{PGL}_{m+1}(k)$, where

$$\xi_m(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)$$

and the second arrow is the canonical projection. Regard $\mathrm{PGL}_n(k)$ as an open subvariety of $\mathbb{P}(\mathrm{M}_n(k))$, and define $\overline{G}(\alpha)$ to be the closure of $\alpha(G)$ in $\mathbb{P}(\mathrm{M}_n(k))$. Since α will be fixed, we shall suppress it and write $\overline{G}$ instead of $\overline{G}(\alpha)$, and we shall identify G with the corresponding open subset of $\overline{G}$.

Notation 2.1 We shall call the identity embedding of $\mathrm{PGL}_n(k)$ into $\mathrm{PGL}_n(k)$ the standard embedding of $\mathrm{PGL}_n(k)$, and the embedding $\mathrm{GL}_n(k) \stackrel{\xi_n}{\to} \mathrm{GL}_{n+1}(k) \to \mathrm{PGL}_{n+1}(k)$, the standard embedding of $\mathrm{GL}_n(k)$.

Given $0 \neq A \in \mathrm{M}_n(k)$, let us denote its image in $\mathbb{P}(\mathrm{M}_n(k))$ by $[A]$. Group multiplication on $G \times G$ extends to a morphism from an open subset of $\overline{G} \times \overline{G}$ into $\overline{G}$: we define $[A][B] = [AB]$ on the set of pairs $([A],[B])$ such that $AB \neq 0$, and we call this morphism **multiplication**. The element $[I]$ is a two-sided identity. The product $[A][B]$ may be defined even when A, B are not invertible matrices. However $[A] \in \overline{G}$ has an inverse in $\overline{G}$ if and only if A is invertible.

Lemma 2.2 *The set O of invertible elements of $\overline{G}$ coincides with G.*

Proof It is clear that O is an algebraic group, with open dense subgroup G. Lemma 7.4 of [4] implies that $G = O$. □

Multiplication on $\overline{G}$ is associative wherever this makes sense, so we define multiplication on any finite product of copies of $\overline{G}$ in the obvious way. We say that $[A_1]\cdots[A_m]$ is defined if $([A_1],\ldots,[A_m])$ is in the domain of the multiplication morphism, that is, if $A_1\cdots A_m \neq 0$.

More generally, let w be a word in letters $\gamma_1,\ldots,\gamma_r$. We write 1 for the empty word. Let $\Sigma = ([A_1],\ldots,[A_r])$ be an r-tuple of elements of $\overline{G}$, and let w be a reduced word in $\gamma_1,\ldots,\gamma_r$. We say that $w([A_1],\ldots,[A_r])$ or $w(\Sigma)$ is defined if:
(i) whenever the letter γ_i^{-1} appears in w, $[A_i]$ is invertible; and
(ii) the element $[B]$ obtained by substituting $[A_i]^{\pm 1}$ for $\gamma_i^{\pm 1}$ in w and then multiplying is defined.
In this case we write $w(\Sigma) = w([A_1],\ldots,[A_r]) = [B]$. For an arbitrary word w, we say that $w([A_1],\ldots,[A_r])$ or $w(\Sigma)$ is defined and equal to $[B]$ if $\mathcal{R}(w)(\Sigma)$ is defined and equal to $[B]$, where $\mathcal{R}(w)$ is the reduced word obtained from w. (We set $1(\Sigma) = [I]$.)

Example 2.3 (a) For any i, $(\gamma_i\gamma_i^{-1})(\Sigma) = 1(\Sigma) = [I]$ is defined.
(b) For any i, $\gamma_i(\Sigma) = [A_i]$ is defined.
(c) For any i, $\gamma_i^{-1}(\Sigma)$ is defined if and only if $[A_i]^{-1}$ is defined if and only if $[A_i] \in G$.
(d) If $[A_i]^{-1}$ is not defined then $(\gamma_i\gamma_j\gamma_i^{-1})(\Sigma)$ is not defined for $j \neq i$, not even if $[A_j] = [I]$.

Observation 2.4 If w is a word then the set

$$\{([A_1],\ldots,[A_r]) \in \overline{G}^r \mid w([A_1],\ldots,[A_r]) \text{ is defined}\}$$

is open in $\overline{G}^r$. Moreover, the function $\Sigma \mapsto w(\Sigma)$ is a morphism from this set to $\overline{G}$.

Lemma 2.5 *Let $\Sigma = ([A_1], \ldots, [A_r]) \in \overline{G}^r$ and let w_1, w_2 be words such that $w_1(\Sigma), w_2(\Sigma)$ and $w_1(\Sigma)w_2(\Sigma)$ are defined. Then $(w_1w_2)(\Sigma)$ is defined and $(w_1w_2)(\Sigma) = w_1(\Sigma)w_2(\Sigma)$.*

Proof Without loss of generality, assume that w_1 and w_2 are reduced. If w_1w_2 is reduced then clearly we are done. Otherwise we can write $w_1 = \widetilde{w}_1\gamma_i^{\pm 1}$ and $w_2 = \gamma_i^{\mp 1}\widetilde{w}_2$ for some i. Then $[A_i]$ is invertible, $\widetilde{w}_1(\Sigma), \widetilde{w}_2(\Sigma)$ are defined and we have $w_1(\Sigma) = \widetilde{w}_1(\Sigma)[A_i]^{\pm 1}$, $w_2(\Sigma) = [A_i]^{\mp 1}\widetilde{w}_2(\Sigma)$, so $w_1(\Sigma)w_2(\Sigma) = \widetilde{w}_1(\Sigma)\widetilde{w}_2(\Sigma)$ and $\mathcal{R}(w_1w_2) = \mathcal{R}(\widetilde{w}_1\widetilde{w}_2)$. The result follows by induction on the lengths of w_1 and w_2. □

As a special case, suppose that $w = \gamma_{m_1}^{\eta_1} \cdots \gamma_{m_s}^{\eta_s}$ is a word (with each $\eta_i \in \{\pm 1\}$), not necessarily reduced, such that $[A_{m_i}]$ is invertible whenever $\eta_i = -1$ and such that $[A_{m_1}]^{\eta_1} \cdots [A_{m_s}]^{\eta_s}$ is defined. Then $w([A_1], \ldots, [A_r])$ is defined and $w([A_1], \ldots, [A_r]) = [A_{m_1}]^{\eta_1} \cdots [A_{m_s}]^{\eta_s}$.

Example 2.6 Let $S = \{\gamma_{m_1}, \ldots, \gamma_{m_s}\} \subset \{\gamma_1, \ldots, \gamma_r\}$. Given a word w in $\gamma_1, \ldots, \gamma_r$, let w_S denote the word in $\gamma_{m_1}, \ldots, \gamma_{m_s}$ obtained from w by deleting all letters $\gamma_i^{\pm 1}$ for all $\gamma_i \notin S$. Suppose that $[A_i] = [I]$ for all i such that $\gamma_i \notin S$. It follows from the special case above that if $w([A_1], \ldots, [A_r])$ is defined, then $w_S([A_1], \ldots, [A_r])$ is also defined and $w([A_1], \ldots, [A_r]) = w_S([A_1], \ldots, [A_r])$.

3 Marked Groups

As noted in §1, the correspondence $F \mapsto \mathrm{R}(F, G)$ is a contravariant functor from the category $\mathfrak{FG}$ of finitely generated groups to the category of affine varieties. We need to keep track of sets of generators as well, so we work with a slightly different category.

Definition 3.1 A **marked group** $\mathcal{F}$ is a pair (F, Δ), where F is a finitely generated group and Δ is a finite set of generators for F. We call Δ a **marking** for F. If (F, Δ) and (F', Δ') are marked groups then a **marked homomorphism** Φ from (F', Δ') to (F, Δ) is a homomorphism $\phi: F' \to F$ such that $\phi(\Delta' \cup \{1\}) \subset \Delta \cup \{1\}$. Define the category $\mathfrak{MG}$ of marked groups to be the category whose objects are the marked groups and whose morphisms are the marked homomorphisms.

Example 3.2 Let $F = F_s$ for some s and let $\gamma_1, \ldots, \gamma_s$ be free generators for F_s. We call $\Delta = \{\gamma_1, \ldots, \gamma_s\}$ the standard marking for F_s.

There is an obvious forgetful functor from $\mathfrak{MG}$ to $\mathfrak{FG}$.

Notation 3.3 If Φ is a marked homomorphism between marked groups then we denote the underlying homomorphism of finitely generated groups by $|\Phi|$. Although Φ is completely determined by $|\Phi|$, nevertheless the distinction between the two is important — see §5 below.

Remark 3.4 Given a pair of marked groups, there are only finitely many marked homomorphisms between them.

If $F' \subset F$, $\Delta' \subset \Delta$ and $|\Phi|$ is inclusion, then we shall refer to Φ as **inclusion** and we shall call (F', Δ') a **marked subgroup** of (F, Δ).

Let $\Phi: (F', \Delta') \to (F, \Delta)$ be a marked homomorphism. We say that Φ is a **marked monomorphism** if the homomorphism $|\Phi|$ is injective. We say that Φ is a **marked epimorphism** if $|\Phi|(\Delta' \cup \{1\}) = \Delta \cup \{1\}$ (this condition of course implies that $|\Phi|$ is an epimorphism). A marked homomorphism is an isomorphism in the category $\mathfrak{MG}$ if and only if it is both a marked monomorphism and a marked epimorphism.

Let (F, Δ) be a marked group and let $N \trianglelefteq F$. Let $Q = F/N$ and let $p: F \to Q$ be the canonical projection. Then $p(\Delta)$ is a finite generating set for Q and there is a marked epimorphism $P: (F, \Delta) \to (Q, p(\Delta))$ induced by p. However, the inclusion of N in F cannot in general be made into a homomorphism of marked groups, even if N is finitely generated. We could have $\Delta \cap N = \emptyset$, for instance, when the only marked subgroup (K, Θ) of (F, Δ) such that $K \subset N$ would be the trivial marked group $(1, \{1\})$.

4 Admissible Representations

Definition 4.1 Let $\mathcal{F} = (F, \Delta)$ be a marked group, with $\Delta = \{\gamma_1, \ldots, \gamma_r\}$. An **admissible representation** of (F, Δ) into $\overline{G}$ is a pair (U, ρ), where $\Delta \subset U \subset F$ and $\rho: U \to \overline{G}$ is a function with the following properties:
(a) for any $\gamma \in F$, $\gamma \in U$ if and only if there is a word w in $\gamma_1, \ldots, \gamma_r$ representing γ such that $w(\rho(\gamma_1), \ldots, \rho(\gamma_r))$ is defined;
(b) for any $\gamma \in U$, for any word w in $\gamma_1, \ldots, \gamma_r$ representing γ such that $w(\rho(\gamma_1), \ldots, \rho(\gamma_r))$ is defined, we have $\rho(\gamma) = w(\rho(\gamma_1), \ldots, \rho(\gamma_r))$.
We say that $\rho(\gamma)$ is defined if and only if $\gamma \in U$. We denote the set of admissible representations by $\mathrm{R}_\Delta(F, \overline{G})$.

Remark 4.2 We shall write $w(\rho)$ as shorthand for $w(\rho(\gamma_1), \ldots, \rho(\gamma_r))$. Note that if $\rho(\gamma)$ is defined, this does not imply that $w(\rho)$ is defined for every word w representing γ, only that $w(\rho)$ is defined for some word w representing γ.

There is a canonical inclusion of $\mathrm{R}(F, G)$ in $\mathrm{R}_\Delta(F, \overline{G})$, given by $\rho \mapsto (F, \rho)$. We denote the image of the trivial representation by $\rho_0^{\mathcal{F}}$.

Lemma 4.3 *(a) Let $(U_1, \rho_1), (U_2, \rho_2) \in \mathrm{R}_\Delta(F, \overline{G})$. If $\rho_1|_\Delta = \rho_2|_\Delta$ then $(U_1, \rho_1) = (U_2, \rho_2)$.*
(b) Let $X_1, \ldots, X_r \in \overline{G}$. There exists an admissible representation (U, ρ) such that $\rho(\gamma_i) = X_i$ for all $1 \leq i \leq r$ if and only if for any words w_1, w_2 in $\gamma_1, \ldots, \gamma_r$ such that w_1 and w_2 represent the same element of F and both $w_1(X_1, \ldots, X_r)$, $w_2(X_1, \ldots, X_r)$ are defined, we have $w_1(X_1, \ldots, X_r) = w_2(X_1, \ldots, X_r)$.

(c) Let $(U,\rho) \in \mathrm{R}_\Delta(F,\overline{G})$ *and let* $\gamma, \widetilde{\gamma} \in U$. *If* $\rho(\gamma)\rho(\widetilde{\gamma})$ *is defined then* $\rho(\gamma\widetilde{\gamma})$ *is defined and*

$$\rho(\gamma\widetilde{\gamma}) = \rho(\gamma)\rho(\widetilde{\gamma}).$$

(Note, however, that even if $\gamma, \widetilde{\gamma}$ *and* $\gamma\widetilde{\gamma}$ *belong to* U, *it does not necessarily follow that the product* $\rho(\gamma)\rho(\widetilde{\gamma})$ *is defined.)*
(d) Let $(U,\rho) \in \mathrm{R}_\Delta(F,\overline{G})$ *and let* $\gamma \in F$. *Suppose there is a word* $w = \gamma_{m_1}^{\eta_1} \cdots \gamma_{m_s}^{\eta_s}$ *representing* γ *(with each* $\eta_i \in \{\pm 1\}$*), not necessarily reduced, such that* $\rho(\gamma_{m_i})$ *is invertible whenever* $\eta_i = -1$ *and such that* $\rho(\gamma_{m_1})^{\eta_1} \cdots \rho(\gamma_{m_s})^{\eta_s}$ *is defined. Then* $\rho(\gamma)$ *is defined and* $\rho(\gamma) = \rho(\gamma_{m_1})^{\eta_1} \ldots \rho(\gamma_{m_s})^{\eta_s}$.

Proof Parts (a) and (b) follow immediately from Definition 4.1, while (c) is a consequence of Lemma 2.5. Part (d) follows from (c) (compare the discussion following Lemma 2.5). □

Convention 4.4 In view of Lemma 4.3, we will regard $\mathrm{R}_\Delta(F,\overline{G})$ as a subset of $\overline{G}^r$, and $\mathrm{R}(F,G)$ as a subset of $\mathrm{R}_\Delta(F,\overline{G})$. Often we will simply write ρ instead of (U,ρ) for an admissible representation.

Remark 4.5 An admissible representation (U,ρ) is completely determined by its values $\rho(\gamma_1),\ldots,\rho(\gamma_r)$. However, the r-tuple $(\rho(\gamma_1),\ldots,\rho(\gamma_r)) \in \overline{G}^r$ does not uniquely determine a subset $\widetilde{U}$ of F and a function $\widetilde{\rho}\colon \widetilde{U} \to F$ satisfying the homomorphism property of Lemma 4.3 (c). For example, there may exist $\widetilde{U} \supset U$ and $\rho_1, \rho_2\colon \widetilde{U} \to \overline{G}$ such that ρ_1, ρ_2 satisfy Lemma 4.3 (c) and $\rho_1|_U = \rho_2|_U$, yet $\rho_1 \neq \rho_2$ — see Example 5.6 below. That is the motivation for Definition 4.1.

The next result shows that $\mathrm{R}_\Delta(F,\overline{G})$ is a compactification of $\mathrm{R}(F,G)$.

Proposition 4.6 *(a) The subset* $\mathrm{R}_\Delta(F,\overline{G})$ *is closed in* $\overline{G}^r$.
(b) The subset $\mathrm{R}(F,G)$ *is open in* $\mathrm{R}_\Delta(F,\overline{G})$.

Proof (a) Let $(X_1,\ldots,X_r)$ lie in the closure of $\mathrm{R}_\Delta(F,\overline{G})$, and suppose that $(X_1,\ldots,X_r) \notin \mathrm{R}_\Delta(F,\overline{G})$. By Lemma 4.3 (b), there exist words w_1, w_2 that represent the same element of F such that $w_1(X_1,\ldots,X_r), w_2(X_1,\ldots,X_r)$ are defined, but $w_1(X_1,\ldots,X_r) \neq w_2(X_1,\ldots,X_r)$. Let O be the set of r-tuples $(Y_1,\ldots,Y_r) \in \overline{G}^r$ such that $w_1(Y_1,\ldots,Y_r), w_2(Y_1,\ldots,Y_r)$ are defined, but $w_1(Y_1,\ldots,Y_r) \neq w_2(Y_1,\ldots,Y_r)$. By Observation 2.4, O is an open neighbourhood of $(X_1,\ldots,X_r)$, hence some element $(Y_1,\ldots,Y_r)$ of $\mathrm{R}_\Delta(F,\overline{G})$ belongs to O. But then we have $w_1(Y_1,\ldots,Y_r) = w_2(Y_1,\ldots,Y_r)$ by Definition 4.1 (b), a contradiction. We conclude that $\mathrm{R}_\Delta(F,\overline{G})$ is closed.
(b) We have $\mathrm{R}(F,G) = \mathrm{R}_\Delta(F,\overline{G}) \cap G^r$, and the result follows. □

Notation 4.7 We denote the closure of $\mathrm{R}(F,G)$ inside $\overline{G}^r$ by $\mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G})$. Example 4.8 (c) below shows that this may be a proper subset of $\mathrm{R}_\Delta(F,\overline{G})$.

Example 4.8 (a) Let $F = F_s$ with the standard marking. Then $\mathrm{R}(F,G) = G^s$ is dense in $\overline{G}^s$, so $\mathrm{R}_\Delta(F,\overline{G}) = \mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G}) = \overline{G}^s$.
(b) Let $F = \langle t, n \mid t^3 = n^2 = 1, ntn^{-1} = t^2\rangle = \mathrm{S}_3$ (the symmetric group on three letters), and let $\Delta = \{n, t\}$. Let $G = \mathrm{PGL}_2(k)$ with the standard embedding. Let $X = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] \in \mathrm{PGL}_2(k)$. Any reduced word $w = w(n,t)$ such that $w([I], X)$ is defined cannot contain the letter t^{-1} and can contain the letter t at most once. Applying Lemma 4.3 (b), we see that $\rho(n) = [I], \rho(t) = X$ defines an admissible representation $\rho \in \mathrm{R}_\Delta(F,\overline{G})$.
(c) Let $F = \mathbb{Z} \oplus \mathbb{Z} = \langle a, b \mid ab = ba\rangle$ and let $\Delta = \{a, b\}$. Let $G = \mathrm{GL}_3(k)$ with the standard embedding. Define $\rho \in \mathrm{R}_\Delta(F,\overline{G})$ by

$$\rho(a) = \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right], \quad \rho(b) = \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]$$

— it is straightforward to check that this is well-defined. Now $\mathrm{SL}_3(k)$ acts on $(\mathrm{M}_4(k)\backslash\{0\}) \times (\mathrm{M}_4(k)\backslash\{0\})$ by conjugation:

$$X.(A,B) = (\xi_3(X)A\xi_3(X^{-1}), \xi_3(X)B\xi_3(X^{-1})).$$

For $A \in \mathrm{M}_4(k)$, let $\mathrm{Z}(A)$ denote the centraliser of A in $\xi_3(\mathrm{SL}_3(k))$. By [10], Lemma 3.7, the function $\aleph\colon \mathrm{M}_4(k)\backslash\{0\} \times \mathrm{M}_4(k)\backslash\{0\} \to \mathbb{N}\cup\{0\}$ given by $\aleph(A,B) = \dim(\mathrm{Z}(A) \cap \mathrm{Z}(B))$ is upper semicontinuous. The $\mathrm{SL}_3(k)$-action commutes with scalar multiplication on each factor, so $\aleph$ descends to a function $\overline{\aleph}\colon \mathbb{P}(\mathrm{M}_4(k)) \times \mathbb{P}(\mathrm{M}_4(k)) \to \mathbb{N}\cup\{0\}$, and one can show that $\overline{\aleph}$ is also upper semicontinuous.

Assume that k has characteristic zero. Then $\mathrm{R}(F,G)$ is irreducible by [11], §2, Proposition 3, so the open subset

$$O = \{\widetilde{\rho} \in \mathrm{R}(F,G) \mid \widetilde{\rho}(a), \widetilde{\rho}(b) \text{ are regular semisimple}\}$$

is dense. For $\widetilde{\rho} \in O$ we have $\overline{\aleph}(\widetilde{\rho}) = 2$, whence $\overline{\aleph}(\widetilde{\rho}) \geq 2$ for all $\widetilde{\rho} \in \mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G})$. But $\overline{\aleph}(\rho) = 1$ by direct computation, so $\rho \notin \mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G})$. Thus $\mathrm{R}_\Delta(F,\overline{G})$ is not a strict compactification of $\mathrm{R}(F,G)$.

Let $\gamma \in F$. The set $\{\rho \in \mathrm{R}_\Delta(F,\overline{G}) \mid \rho(\gamma) \text{ is defined}\}$ is open by Observation 2.4. Define a function ϵ_γ from this open set to $\overline{G}$ by $\epsilon_\gamma(\rho) = \rho(\gamma)$. If $\rho(\gamma)$ is defined then there is a reduced word w in $\gamma_1, \ldots, \gamma_r$ such that w represents γ and the function $\widetilde{\rho} \mapsto w(\widetilde{\rho}(\gamma_1), \ldots, \widetilde{\rho}(\gamma_r))$ is well-defined and equal to ϵ_γ on an open neighbourhood of ρ. We deduce that ϵ_γ is a morphism of varieties. The morphisms ϵ_{γ_i} are defined everywhere on $\mathrm{R}_\Delta(F,\overline{G})$ and the canonical embedding of $\mathrm{R}_\Delta(F,\overline{G})$ in $\overline{G}^r$ (see Convention 4.4) is just the product $\epsilon_{\gamma_1} \times \cdots \times \epsilon_{\gamma_r}$.

Let $X_1, \ldots, X_r \in \overline{G}$. Except for in simple cases, it is impractical to check whether the prescription $\rho(\gamma_i) = X_i$ gives an admissible representation $\rho \in$

$\mathrm{R}_\Delta(F,\overline{G})$ by directly verifying that the hypotheses of Lemma 4.3 (b) hold. An easier way to obtain admissible representations is by taking limits.

Let C be a smooth irreducible quasiprojective curve and let $x_0 \in C$. If $f: C\backslash\{x_0\} \to V$ is a morphism of varieties, we say that $\lim_{x\to x_0} f(x)$ exists and equals v if f extends to a morphism $f': C \to V$ such that $f'(x_0) = v$. If a limit exists then it is unique. The limit always exists if V is projective ([3], Chapter I, Proposition 6.8). Thus, given a morphism $C\backslash\{x_0\} \to \mathrm{R}_\Delta(F,\overline{G})$, $x \mapsto \rho_x$, the limit $\lim_{x\to x_0}\rho_x$ also belongs to $\mathrm{R}_\Delta(F,\overline{G})$. Moreover, if $\rho_x \in \mathrm{R}(F,G)$ for each $x \in C\backslash\{x_0\}$, then $\lim_{x\to x_0}\rho_x \in \mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G})$.

Example 4.9 Let $F, \Delta, G, \overline{G}, \rho$ be as in Example 4.8 (b). Assume that k has characteristic two. For $x \in k^*$, define $\rho_x \in \mathrm{R}(F,G)$ by

$$\rho_x(n) = \left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right],\ \rho_x(t) = \left[\begin{pmatrix} \zeta & 0 \\ x^{-1}(\zeta+\zeta^2) & \zeta^2 \end{pmatrix}\right] = \left[\begin{pmatrix} x\zeta & 0 \\ \zeta+\zeta^2 & x\zeta^2 \end{pmatrix}\right],$$

where ζ is a primitive cube root of unity. Then $\lim_{x\to 0}\rho_x = \rho$, and we deduce that $\rho \in \mathrm{R}^{\mathrm{s}}_\Delta(F,\overline{G})$.

Remark 4.10 Let $\rho = \lim_{x\to x_0}\rho_x$ for some map $x \mapsto \rho_x$ as above. Let $\gamma \in F$. If $\rho_y(\gamma)$ is defined for some $y \in C\backslash\{x_0\}$ then the morphism $x \mapsto \rho_x(\gamma)$ is defined on an open neighbourhood C' of y in $C\backslash\{x_0\}$, and we can form the limit $\lim_{x\to x_0}\rho_x(\gamma)$. In particular, if $\rho(\gamma)$ is defined then $\lim_{x\to x_0}\rho_x(\gamma) = \lim_{x\to x_0}\epsilon_\gamma(\rho_x)$ exists because ϵ_γ is a morphism, and the limit is equal to $\rho(\gamma)$. However, even when $\rho(\gamma)$ is not defined, $\lim_{x\to x_0}\rho_x(\gamma)$ will exist if $\rho_x(\gamma)$ is defined for some $x \in C\backslash\{x_0\}$ (see Example 5.6).

5 Functoriality

Proposition 5.1 *Let $\Phi: (F',\Delta') \to (F,\Delta)$ be a homomorphism of marked groups, where $\Delta = \{\gamma_1,\ldots,\gamma_r\}$ and $\Delta' = \{\gamma'_1,\ldots,\gamma'_t\}$. Let $(U,\rho) \in \mathrm{R}_\Delta(F,\overline{G})$. The t-tuple $(\rho'(\gamma'_1),\ldots,\rho'(\gamma'_t))$ defined by $\rho'(\gamma'_i) = \rho(|\Phi|(\gamma'_i))$ determines a unique admissible representation $(U',\rho') \in \mathrm{R}_{\Delta'}(F',\overline{G})$. We have $|\Phi|(U') \subset U$, with equality if Φ is a marked epimorphism, and*

$$\rho' = \rho \circ |\Phi|. \tag{1}$$

Proof Let $w', \widetilde{w}'$ be words in $\gamma'_1,\ldots,\gamma'_t$ such that $w', \widetilde{w}'$ both represent the same element $\gamma' \in F'$ and $w'(\rho'), \widetilde{w}'(\rho')$ are defined. Define a word w in $\gamma_1,\ldots,\gamma_r$ as follows: replace every occurrence of $(\gamma'_i)^{\pm1}$ in w' with $\gamma_{m_i}^{\pm1}$, where $\gamma_{m_i} = |\Phi|(\gamma_i)$, and define $\widetilde{w}$ from $\widetilde{w}'$ similarly. Then $w, \widetilde{w}$ both represent $|\Phi|(\gamma') \in F$, $w(\rho)$ and $\widetilde{w}(\rho)$ are both defined, and we have $w(\rho) = w'(\rho')$ and $\widetilde{w}(\rho) = \widetilde{w}'(\rho')$. But $w(\rho) = \widetilde{w}(\rho) = \rho(|\Phi|(\gamma'))$, so $w'(\rho') = \widetilde{w}'(\rho')$. Therefore ρ' is an admissible representation by Lemma 4.3 (b). Uniqueness follows from Lemma 4.3 (a), and Equation 1 is a consequence of Lemma 4.3 (d). Finally, if Φ is a marked epimorphism then every word w in $\gamma_1, \ldots, \gamma_r$ can be obtained from some word w' in $\gamma'_1, \ldots, \gamma'_t$ in the way described above, and it follows easily that $|\Phi|(U') = U$. □

Definition 5.2 Define $\Phi^{\#}: \mathrm{R}_\Delta(F, \overline{G}) \to \mathrm{R}_{\Delta'}(F', \overline{G})$ by $\Phi^{\#}((U, \rho)) = (U', \rho')$, where (U', ρ') is as given above.

In terms of our canonical embeddings in projective varieties (see Convention 4.4), $\Phi^{\#}$ is the restriction to $\mathrm{R}_\Delta(F, \overline{G})$ of a product of projections onto factors of $\overline{G}^r$, so $\Phi^{\#}$ is a morphism. It is easily verified that $(F, \Delta) \mapsto \mathrm{R}_\Delta(F, \overline{G})$, $\Phi \mapsto \Phi^{\#}$ is a contravariant functor from $\mathfrak{MG}$ to the category of projective varieties, and the restriction of $\Phi^{\#}$ to $\mathrm{R}(F, G)$ is equal to $\phi^{\#}: \mathrm{R}(F, G) \to \mathrm{R}(F', G)$ from §1.

Example 5.3 If $\Phi: (F', \Delta') \to (F, \Delta)$ is a marked epimorphism then by considering the canonical embeddings of $\mathrm{R}_{\Delta'}(F', \overline{G})$ and $\mathrm{R}_\Delta(F, \overline{G})$, we see that $\Phi^{\#}$ is a closed embedding. In particular, if F' is free on $\gamma'_1, \ldots, \gamma'_t$ and $\Delta' = \{\gamma'_1, \ldots, \gamma'_t\}$ then, identifying $\mathrm{R}_{\Delta'}(F', \overline{G})$ with $\overline{G}^t$ as in Example 4.8 (a), we have $\Phi^{\#} = \epsilon_{|\Phi|(\gamma'_1)} \times \cdots \times \epsilon_{|\Phi|(\gamma'_t)}$.

When comparing different compactifications of $\mathrm{R}(F, G)$, it is helpful to consider inclusions $J: (F, \Delta') \to (F, \Delta)$. If Δ' is properly contained in Δ then $J^{\#}$ need not be an isomorphism (see Example 5.6 below). Here is a criterion for $J^{\#}$ to be an isomorphism near a point.

Lemma 5.4 *Let $\Phi: (F, \Delta') \to (F, \Delta)$ be an inclusion. Set*

$$O = \{\rho' \in \mathrm{R}_{\Delta'}(F, \overline{G}) \mid \rho'(\gamma) \text{ is defined for all } \gamma \in \Delta \backslash \Delta'\}.$$

Then O is open and $\Phi^{\#}|_{(\Phi^{\#})^{-1}(O)}: (\Phi^{\#})^{-1}(O) \to O$ is an isomorphism.

Proof Write $\Delta \backslash \Delta' = \{\gamma_1, \ldots, \gamma_s\}$. Let $\rho' \in O$. There are words $w_1, \ldots, w_s$ in the generators from Δ' such that w_i represents γ_i and $w_i(\rho')$ is defined. Then $w_1(\rho), \ldots, w_s(\rho)$ are defined for ρ in an open neighbourhood O' of ρ'. Define $\psi: O' \to \mathrm{R}_\Delta(F, \overline{G})$ by $\psi(\rho)(\gamma') = \rho(\gamma')$ for $\gamma' \in \Delta'$, and $\psi(\rho)(\gamma_i) = w_i(\rho)$. It is simple to check that ψ is well-defined, that $\psi(O') = (\Phi^{\#})^{-1}(O')$, that $O' \subset O$ and that $\psi: O' \to (\Phi^{\#})^{-1}(O')$ is a morphism with inverse $\Phi^{\#}|_{(\Phi^{\#})^{-1}(O')}$. The result follows. □

As an immediate corollary, we have

Corollary 5.5 *Let (F, Δ) be a marked group, let $\gamma \in F$ and let $\Phi: (F, \Delta) \to (F, \Delta \cup \{\gamma\})$ be inclusion. Given $\rho \in \mathrm{R}_\Delta(F, \overline{G})$, if $(\Phi^{\#})^{-1}(\rho)$ contains more than one point then $\rho(\gamma)$ is not defined.* □

Example 5.6 Let $F = F_2$, with free generators γ_1, γ_2, and let $G = \mathrm{PGL}_2(k)$ with the standard embedding. For any $n \in \mathbb{N}$, define $\Delta_n = \{\gamma_1, \gamma_2, \gamma_1\gamma_2\gamma_1^{-1}, \gamma_1^2\gamma_2\gamma_1^{-2}, \ldots, \gamma_1^n\gamma_2\gamma_1^{-n}\}$, and for $N \in \mathbb{N}, x \in k^*$, define $\rho_x^{(N)} \in \mathrm{R}(F, G)$ by

$$\rho_x^{(N)}(\gamma_1) = \left[\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}\right], \; \rho_x^{(N)}(\gamma_2) = \left[\begin{pmatrix} 1 & x^{2N-1} \\ 0 & 1 \end{pmatrix}\right].$$

Let $\Phi_n\colon (F,\Delta_n) \to (F,\Delta_{n+1})$ be inclusion.

Regarding each $\rho_x^{(N)}$ as an element of $\mathrm{R}_{\Delta_n}(F,\overline{G})$, define $\sigma_n, \tau_n \in \mathrm{R}^{\mathrm{s}}_{\Delta_n}(F,\overline{G})$ to be $\lim_{x\to 0}\rho_x^{(n)}$ and $\lim_{x\to 0}\rho_x^{(n+1)}$ respectively. Then $\sigma_n \neq \tau_n$ — for

$$\sigma_n(\gamma_1^n\gamma_2\gamma_1^{-n}) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \neq [I] = \tau_n(\gamma_1^n\gamma_2\gamma_1^{-n})$$

— but $\Phi_n^{\#}(\sigma_{n+1}) = \Phi_n^{\#}(\tau_{n+1}) = \tau_n$. We conclude from Corollary 5.5 that $\tau_n(\gamma_1^{n+1}\gamma_2\gamma_1^{-(n+1)})$ is undefined, even though $\lim_{x\to 0}\rho_x^{(n+1)}(\gamma_1^{n+1}\gamma_2\gamma_1^{-(n+1)}) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$ exists. Moreover, none of the compositions $\Phi_n^{\#}\circ\Phi_{n+1}^{\#}\circ\cdots\circ\Phi_{n+m-1}^{\#}$ gives an isomorphism from $\mathrm{R}^{\mathrm{s}}_{\Delta_{n+m}}(F,\overline{G})$ to $\mathrm{R}^{\mathrm{s}}_{\Delta_n}(F,\overline{G})$ for any $n,m\in\mathbb{N}$, so the strict compactifications $\mathrm{R}^{\mathrm{s}}_{\Delta_n}(F,\overline{G})$ are pairwise nonisomorphic. This implies that the compactifications $\mathrm{R}_{\Delta_n}(F,\overline{G})$ are also pairwise nonisomorphic.

Example 5.6 shows that the compactifications arising from different generating sets may be distinct. Now we see that this problem cannot be avoided.

Theorem 5.7 *There does not exist a contravariant functor* $F \mapsto \widetilde{\mathrm{R}}(F,G)$, $\phi \mapsto \widetilde{\phi}^{\#}$ *from the category* $\mathfrak{FG}$ *to the category of projective varieties such that:*
(i) for every finitely generated group F, $\widetilde{\mathrm{R}}(F,G)$ *is a compactification of* $\mathrm{R}(F,G)$*; and*
(ii) for every homomorphism $\phi\colon F' \to F$ *of finitely generated groups, the restriction to* $\mathrm{R}(F,G)$ *of the morphism* $\widetilde{\phi}^{\#}\colon \widetilde{\mathrm{R}}(F,G) \to \widetilde{\mathrm{R}}(F',G)$ *equals* $\phi^{\#}$.

Proof Suppose that such a functor exists. Let $F, G, \gamma_1, \gamma_2, \Delta_n, \rho_x^{(N)}$ be as in Example 5.6. Given $\gamma\in F$, let $i_\gamma\colon \mathbb{Z}\to F$ be the homomorphism that takes $1\in\mathbb{Z}$ to γ. By hypothesis, we have a morphism $\widetilde{i_\gamma}^{\#}\colon \widetilde{\mathrm{R}}(F,G) \to \widetilde{\mathrm{R}}(\mathbb{Z},G)$ that extends $i_\gamma^{\#}$. Set $D_n = \bigcap_{\gamma\in\Delta_n\setminus\{\gamma_1\}}(\widetilde{i_\gamma}^{\#})^{-1}(\rho_0^{\mathbb{Z}}) \subset \widetilde{\mathrm{R}}(F,G)$. It is clear that $D_1 \supset D_2 \supset \cdots$.

Define $\beta_n = \lim_{x\to 0}\rho_x^{(n+1)} \in \widetilde{\mathrm{R}}(F,G)$. Then

$$\begin{aligned}\widetilde{i}^{\#}_{\gamma_1^m\gamma_2\gamma_1^{-m}}(\beta_n) &= \lim_{x\to 0}\widetilde{i}^{\#}_{\gamma_1^m\gamma_2\gamma_1^{-m}}(\rho_x^{(n)}) = \lim_{x\to 0} i^{\#}_{\gamma_1^m\gamma_2\gamma_1^{-m}}(\rho_x^{(n)})\\ &= \lim_{x\to 0}\rho_x^{(n)}(\gamma_1^m\gamma_2\gamma_1^{-m}) = I\end{aligned}$$

for all m such that $1\le m\le n$. However, we have $\widetilde{i}^{\#}_{\gamma_1^{n+1}\gamma_2\gamma_1^{-(n+1)}}(\beta_n) = \lim_{x\to 0}\rho_x^{(n)}(\gamma_1^{n+1}\gamma_2\gamma_1^{-(n+1)})$ which does not equal I, so $\beta_n \in D_n\setminus D_{n+1}$. But this implies that the sequence $D_1\supset D_2\supset\cdots$ is strictly decreasing, which contradicts the descending chain condition on closed subsets of $\widetilde{\mathrm{R}}(F,G)$. □

Remark 5.8 If we restrict ourselves to the category of finite groups then we can find a functor with the properties given in Theorem 5.7. We set $\widetilde{\mathrm{R}}(F,G) =$

$\mathrm{R}_F(F,\overline{G})$. Then any homomorphism $\phi: F' \to F$ of finite groups gives rise to a marked homomorphism $\Phi: (F',F') \to (F,F)$ such that $|\Phi| = \phi$, and we define $\widetilde{\phi}^{\#} = \Phi^{\#}$.

Remark 5.9 Theorem 5.7 says that we cannot make a single uniform choice of compactification $\widetilde{\mathrm{R}}(F,G)$ for every F and preserve functoriality. However, given a single pair of finitely generated groups F, F' and a homomorphism $\phi: F' \to F$, we can extend $\phi^{\#}$ to a morphism $\Phi^{\#}: \mathrm{R}_\Delta(F,\overline{G}) \to \mathrm{R}_{\Delta'}(F',\overline{G})$ for an appropriate choice of markings Δ, Δ'.

6 Approximate Kernels

Let $p: F \to Q$ be an epimorphism of finitely generated groups, with kernel N.

Lemma 6.1 *There exists an approximate G-kernel of p.*

Proof Suppose not. Then we can find finitely generated subsets $K_1 \subset K_2 \subset \cdots$ of N such that the chain $j_1^{-1}(\rho_0^{K_1}) \supset j_2^{-1}(\rho_0^{K_2}) \supset \cdots$ is strictly decreasing, where j_i is the inclusion of K_i in F. But this contradicts the descending chain condition on closed subsets of $\mathrm{R}(F,G)$. □

Remark 6.2 If N is the normal closure of K in F, then for any G, K is an approximate G-kernel of p.

We want to describe the image of $P^{\#}$ for compactified representation varieties in a similar way.

Lemma 6.3 *Let $P: (F,\Delta) \to (Q,\Xi)$ be a marked epimorphism. Let $(U,\rho) \in \mathrm{R}_\Delta(F,\overline{G})$. Then ρ is in the image of $P^{\#}$ if and only if:*

($*$) *for all $\gamma, \widetilde{\gamma} \in U$ such that $|P|(\gamma) = |P|(\widetilde{\gamma})$, we have $\rho(\gamma) = \rho(\widetilde{\gamma})$.*

Proof If ($*$) holds then we have a well-defined function $\sigma: |P|(U) \to \overline{G}$ given by $\sigma(|P|(\gamma)) = \rho(\gamma)$ for $\gamma \in U$. It is easily checked that $(|P|(U),\sigma)$ is an admissible representation and that $\rho = P^{\#}(\sigma)$ (compare the proof of Proposition 5.1). The converse follows from Equation 1. □

Definition 6.4 Let $P: (F,\Delta) \to (Q,\Xi)$ be a marked epimorphism. Let $\mathcal{K} = (K,\Theta)$ be a marked group with $K \subset \ker|P|$ and $\Theta \subset \Delta$, and let $J: (K,\Theta) \to (F,\Delta)$ be inclusion. We say that $\mathcal{K}$ is an **approximate $\overline{G}$-kernel** of P if $P^{\#}(\mathrm{R}_\Xi(Q,\overline{G})) = (J^{\#})^{-1}(\rho_0^{\mathcal{K}})$. (Note that $P^{\#}(\mathrm{R}_\Xi(Q,\overline{G}))$ is always contained in $(J^{\#})^{-1}(\rho_0^{\mathcal{K}})$.)

If instead of P we are given an epimorphism $p: F \to Q$ then we take P to be the obvious marked epimorphism. If just a normal subgroup N of F is given then we take p to be the canonical epimorphism $F \to F/N$. If just K is given then we take N to be the normal closure of K in F, and we say that $\mathcal{K}$ is an approximate kernel of (F,Δ).

Proposition 6.5 *Let $P: (F, \Delta) \to (Q, \Xi)$ be a marked epimorphism, and let $\mathcal{K} = (K, \Theta)$ be a marked subgroup of (F, Δ) with $K \subset \ker|P|$.*
(a) If (K, Θ) is an approximate $\overline{G}$-kernel of P, then K is an approximate G-kernel of $|P|$.
(b) If $K = \ker|P|$ then (K, Θ) is an approximate $\overline{G}$-kernel of P.

Proof (a) Let $\rho \in \mathrm{R}_\Xi(Q, \overline{G})$. Since $|P|$ is surjective, ρ belongs to $\mathrm{R}(Q, G)$ if and only if $P^\#(\rho)$ belongs to $\mathrm{R}(F, G)$. Part (a) now follows.
(b) Let $J: (K, \Theta) \to (F, \Delta)$ be inclusion. Choose any $(U, \rho) \in \mathrm{R}_\Delta(F, \overline{G})$ such that $J^\#(\rho) = \rho_0^{\mathcal{K}}$. If $\gamma, \tilde{\gamma} \in U$ such that $|P|(\gamma) = |P|(\tilde{\gamma})$ then we have $\tilde{\gamma} = \alpha\gamma$ for some $\alpha \in K$. Since $\rho(\alpha) = [I]$, $\rho(\alpha)\rho(\gamma)$ is defined and $\rho(\tilde{\gamma}) = [I] \cdot \rho(\gamma) = \rho(\gamma)$ by Lemma 4.3 (c). Lemma 6.3 now implies that ρ lies in the image of $P^\#$, as required. □

Example 6.6 (a) Let $s \in \mathbb{N}$ and let Δ be the standard marking for F_s. Given $\Theta \subset \Delta$, set $\mathcal{K} = (\langle\Theta\rangle, \Theta)$. It is clear that $\mathcal{K}$ is an approximate $\overline{G}$-kernel of (F_s, Δ).
(b) Let $F, \Delta, G, \overline{G}, \rho$ be as in Example 4.8 (b). Let P be the marked epimorphism from (F, Δ) onto the trivial marked group $(1, \{1\})$. Let $\mathcal{K} = (\langle n\rangle, \{n\})$ and let N be the normal closure of K in F. Then $N = F = \ker|P|$; but $\mathcal{K}$ is not an approximate $\overline{G}$-kernel of P, for ρ is nontrivial yet $\rho(n) = [I]$.

Given an extension $1 \to N \to F \to Q \to 1$ of groups with F finitely generated and given a marking Δ for F, one shouldn't expect an approximate $\overline{G}$-kernel to exist, for the reasons given at the end of §3. This leads us to broaden our definition of approximate kernel.

Lemma 6.7 *Let $p: F \to Q$ be an epimorphism, let Δ_1, Δ_2 be markings for F and let (K, Θ_1), (K, Θ_2) be marked subgroups of (F, Δ_1), (F, Δ_2) respectively, with $K \subset \ker p$. Suppose that $\Delta_1 \backslash \Theta_1 = \Delta_2 \backslash \Theta_2$. Then (K, Θ_1) is an approximate $\overline{G}$-kernel of (F, Δ_1) if and only if (K, Θ_2) is an approximate $\overline{G}$-kernel of (F, Δ_2).*

Proof Let $P_1: (F, \Delta_1) \to (Q, p(\Delta_1))$ and $P_2: (F, \Delta_2) \to (Q, p(\Delta_2))$ be the marked epimorphisms induced by p. Set $S = \Delta_1 \backslash \Theta_1 = \Delta_2 \backslash \Theta_2$. Let $\rho_1 \in \mathrm{R}_{\Delta_1}(F, \overline{G})$ such that $\rho_1(\gamma) = [I]$ for all $\gamma \in \Theta_1$. For each $\gamma \in \Theta_1$, choose a word v_γ in the generators from Θ_2 such that v_γ represents γ. Given a word w_1 in the generators from Δ_1, we obtain a word w_2 in the generators from Δ_2 by replacing every occurrence of $\gamma^{\pm 1}$ in w_1 by $v_\gamma^{\pm 1}$, for all $\gamma \in \Theta_1$. Then w_1, w_2 represent the same element of F and $(w_1)_S = (w_2)_S$, where $(w_1)_S, (w_2)_S$ are defined as in Example 2.6.

Define ρ_2 by $\rho_2(\gamma) = [I]$ for $\gamma \in \Theta_2$ and $\rho_2(\gamma) = \rho_1(\gamma)$ for $\gamma \in S$. If $w_1(\rho_1)$ is defined then $w_2(\rho_2)$ is defined and $w_2(\rho_2) = (w_2)_S(\rho_2) = (w_1)_S(\rho_1) = w_1(\rho_1)$. It is now straightforward to verify that ρ_2 belongs to $\mathrm{R}_{\Delta_2}(F, \overline{G})$ and that moreover, $\rho_1 \in \operatorname{Im} P_1^\#$ if and only if $\rho_2 \in \operatorname{Im} P_2^\#$.

By symmetry, we can interchange the roles of (F, Δ_1) and (F, Δ_2), and the result follows. □

Definition 6.8 We say that a subgroup K of $\ker p$ is an **approximate $\overline{G}$-kernel** of (F, Δ) if

$(*)$ (K, Θ) is an approximate $\overline{G}$-kernel of $(F, \Delta \cup \Theta)$ in the sense of Definition 6.4,

for some marking Θ. In view of Lemma 6.7, the question of whether $(*)$ holds is independent of the choice of Θ.

Theorem 6.9 *Let $P\colon (F, \Delta) \to (Q, \Xi)$ be a marked epimorphism. There exists a finitely generated subgroup K of $\ker |P|$ such that K is an approximate kernel of P.*

Proof Given a marked group $\widetilde{\mathcal{K}} = (\widetilde{K}, \widetilde{\Theta})$ with $\widetilde{K} \subset \ker |P|$, let $J_{\widetilde{\mathcal{K}}}\colon (\widetilde{K}, \widetilde{\Theta}) \to (F, \Delta \cup \widetilde{\Theta})$ and $L_{\widetilde{\mathcal{K}}}\colon (F, \Delta) \to (F, \Delta \cup \widetilde{\Theta})$ be the inclusions. Define a closed subset $V_{\widetilde{\mathcal{K}}}$ of $\mathrm{R}_\Delta(F, \overline{G})$ by $V_{\widetilde{\mathcal{K}}} = L^\#_{\widetilde{\mathcal{K}}}((J^\#_{\widetilde{\mathcal{K}}})^{-1}(\rho_0^{\widetilde{\mathcal{K}}}))$. Since the variety $\mathrm{R}_\Delta(F, \overline{G})$ is a noetherian topological space, we can choose $\mathcal{K} = (K, \Theta)$ such that $V_\mathcal{K}$ is minimal. We show that $\mathcal{K}$ is an approximate $\overline{G}$-kernel of $(F, \Delta \cup \Theta)$.

Let $(U, \rho) \in \mathrm{R}_{\Delta \cup \Theta}(F, \overline{G})$ such that $(J^\#_\mathcal{K})^{-1}(\rho) = \rho_0^\mathcal{K}$. Choose any $\gamma, \delta \in U$ such that $|P|(\gamma) = |P|(\delta)$. Then $\delta = \alpha\gamma$ for some $\alpha \in \ker |P|$. Set $\widetilde{\mathcal{K}} = (\widetilde{K}, \widetilde{\Theta})$, where $\widetilde{K} = \langle K \cup \{\alpha\}\rangle$ and $\widetilde{\Theta} = \Theta \cup \{\alpha\}$, and let $M\colon (F, \Delta \cup \Theta) \to (F, \Delta \cup \widetilde{\Theta})$ be inclusion. Then $V_{\widetilde{\mathcal{K}}} \subset V_\mathcal{K}$. Thus $V_{\widetilde{\mathcal{K}}} = V_\mathcal{K}$ by choice of $\mathcal{K}$, so we can find $(\widetilde{U}, \widetilde{\rho}) \in \mathrm{R}_{\Delta \cup \widetilde{\Theta}}(F, \overline{G})$ such that $L^\#_{\widetilde{\mathcal{K}}}(\widetilde{\rho}) = L^\#_\mathcal{K}(\rho)$ and $(J^\#_{\widetilde{\mathcal{K}}})^{-1}(\widetilde{\rho}) = \rho_0^{\widetilde{\mathcal{K}}}$. It is easily checked that $M^\#(\widetilde{\rho}) = \rho$, whence $\gamma, \delta \in \widetilde{U}$ and $\widetilde{\rho}(\gamma) = \rho(\gamma)$, $\widetilde{\rho}(\delta) = \rho(\delta)$. But $\widetilde{\rho}(\alpha) = [I]$, so

$$\rho(\delta) = \widetilde{\rho}(\delta) = [I] \cdot \widetilde{\rho}(\gamma) = \widetilde{\rho}(\gamma) = \rho(\gamma).$$

By Lemma 6.3, we are done. □

The next example illustrates that we cannot choose a K that works for all choices of marking Δ.

Example 6.10 Let $F = F_2$ with free generators γ_1, γ_2 and let $\Delta = \{\gamma_1, \gamma_2\}$. Let $Q = \mathbb{Z}$ with generator γ_1 and let $\Xi = \{\gamma_1\}$. Define $P\colon (F, \Delta) \to (Q, \Xi)$ by $|P|(\gamma_1) = \gamma_1$, $|P|(\gamma_2) = 1$. Let $G = \mathrm{PGL}_2(k)$ with the standard embedding.

Now choose any finitely generated subgroup K of $\ker |P|$ and any marking Θ for K. Since $\ker |P|$ is not finitely generated ([7], Theorem 2), we can pick $\delta \in \ker |P|$ such that $\delta \notin \langle K \cup \{\gamma_2\}\rangle$. Let $\Delta_\delta = \{\gamma_1, \gamma_2, \delta\}$. We show that (K, Θ) is not an approximate $\overline{G}$-kernel of $(F, \Delta_\delta \cup \Theta)$.

Define $\rho(\gamma)$ for $\gamma \in \Delta_\delta \cup \Theta$ by

$$\rho(\gamma) = [I] \text{ for } \gamma \in \Theta \cup \{\gamma_2\}, \ \rho(\gamma_1) = \rho(\delta) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right].$$

We check that ρ gives an admissible representation. If w is any reduced word in the generators from $\Delta_\delta \cup \Theta$ such that $w(\rho)$ is defined, then neither of the letters

γ_1^{-1} and δ^{-1} can appear in w. There are three possibilities:
(i) neither of γ_1, δ appears in w (in which case $w(\rho) = [I]$);
(ii) γ_1 appears exactly once in w and δ does not appear (in which case $w(\rho) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$);
(iii) δ appears exactly once in w and γ_1 does not appear (in which case $w(\rho) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$).
Let $w, \widetilde{w}$ be two reduced words in the generators from $\Delta_\delta \cup \Theta$ such that $w(\rho), \widetilde{w}(\rho)$ are defined and $w, \widetilde{w}$ represent the same element of F. We must show that $w(\rho) = \widetilde{w}(\rho)$. Clearly we need only consider the case when (i) holds for w and (iii) holds for $\widetilde{w}$: say $\widetilde{w} = u\delta v$, for some words u, v in the generators from $\Theta \cup \{\gamma_2\}$. But then δ and $u^{-1}wv^{-1}$ represent the same element of F, which is impossible because $u^{-1}wv^{-1}$ represents an element of $\langle K \cup \{\gamma_2\}\rangle$. Thus ρ is well-defined.

We see that $\rho(\gamma) = [I]$ for $\gamma \in \Theta$, but $\rho(\delta) \neq [I]$, so (K, Θ) is not an approximate $\overline{G}$-kernel of $(F, \Delta_\delta \cup \Theta)$, as claimed. It follows that K is not an approximate $\overline{G}$-kernel of (F, Δ_δ).

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