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Integral rigid sets and periods of nonexpansive maps

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ABSTRACT

For ℓ_1-norm and sup-norm nonexpansive maps it is known that bounded orbits approach periodic orbits. Moreover the minimal period of a periodic point of such a map has an a priori upper bound that only depends on the dimension and the given norm. We shall show that the question whether for a given positive integer p there exists an ℓ_1-norm (or sup-norm) nonexpansive map $f : D_f \to D_f \subset \mathbb{R}^n$ with a periodic point of minimal period p can be answered in finite time

1 INTRODUCTION

Let $(V, \|\cdot\|)$ be a Banach space. A map $f : D \subset V \to V$ is called *nonexpansive with respect to* $\|\cdot\|$, if

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \text{for every } x, y \in D.$$

A point $x \in D$ is called *a periodic point of* f *with minimal period* p if $f^p(x) = x$ and $f^j(x) \neq x$ for $1 \leq j < p$. Here f^j denotes the composition of f with itself j times. The sequence

$$x, f(x), f^2(x), f^3(x), \ldots$$

will be called the *orbit of x under f*.
As usual, the ℓ_1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$ is defined by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

If $D \subset \mathbb{R}^n$ is closed, $g : D \to D$ is an ℓ_1-norm nonexpansive map, and there exists $z \in D$ with bounded orbit under g, then for each $x \in D$, there exists a positive integer $p = p_x$ and a point $\xi = \xi_x \in D$ such that

$$\lim_{i \to \infty} g^{pi}(x) = \xi,$$

where ξ is a periodic point of g with minimal period p. Moreover, the number p_x is bounded by $n!2^m$, where $m = 2^n$.

The original proof of this result was given by Akcoglu and Krengel [1]. Their arguments, however, did not provide an upper bound on the integer p_x, $x \in D$. The upper bound of $n!2^m$, where $m = 2^n$, has been obtained by Misiurewicz [5], but this estimate is expected to be far from sharp. The question of finding an optimal upper bound on the integer p_x appears to be very difficult.

This result implies that for a map $f : D_f \to D_f$, $D_f \subset \mathbb{R}^n$, which is ℓ_1-norm nonexpansive, the periodic points play a key role in understanding the behaviour of $f^j(x)$ as $j \to \infty$. Therefore it is an interesting problem to determine for each $n \in \mathbb{N}$ the finite set.

$$\tilde{R}(n) = \{p \in \mathbb{N} \mid \exists f : D_f \to D_f,\ D_f \subset \mathbb{R}^n, \text{such that } f \text{ is } \ell_1\text{-norm}$$
$$\text{nonexpansive and } f \text{ has a periodic point of minimal period } p\}.$$

This problem was raised in [8, Remark 4.1], and from this paper we took the notation.

It is known that an ℓ_1-norm nonexpansive map $f : D_f \to D_f$ may not have an ℓ_1-norm nonexpansive extension $F : \mathbb{R}^n \to \mathbb{R}^n$. Therefore it is also important to study the set of possible minimal periods for fixed domains D in $\mathbb{R}^n$. Some special domains are: the whole space $\mathbb{R}^n$, the positive cone

$$\mathbb{K}^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, \text{ for } 1 \leq i \leq n\}$$

and the unit simplex

$$\Delta^n = \{x \in \mathbb{K}^n \mid \sum_{i=1}^{n} x_i = 1\}.$$

In this paper, however, we will look at the most general set $\tilde{R}(n)$, which is the union of possible minimal periods taken over all domains D in $\mathbb{R}^n$.

We remark that the set

$$R(n) = \{p \in \mathbb{N} \mid \exists f : \mathbb{R}^n \to \mathbb{R}^n \text{ such that } f \text{ is } \ell_1\text{-norm nonexpansive}$$
$$\text{and } f \text{ has a periodic point of minimal period } p\}$$

is contained in $\tilde{R}(n)$, and it is expected (see [8, Section 4]) that $R(n)$ is strictly smaller than $\tilde{R}(n)$, for $n \geq 3$.

Furthermore, if $f : \mathbb{K}^n \to \mathbb{K}^n$ is an ℓ_1-norm nonexpansive map and $f(0) = 0$, then sharp results on the possible minimal periods of periodic points of f were obtained by Nussbaum, Scheutzow and Verduyn Lunel (see [7], [8] and [9]). Moreover, it follows from the proofs that the set of positive integers p such that there exists an ℓ_1-norm nonexpansive map $f : \mathbb{K}^n \to \mathbb{K}^n$ with $f(0) = 0$, which has a periodic point of minimal period p, can be determined by considering only periods p that come from periodic orbits that are contained in $\mathbb{Z}^n$.

For the general set of periods $\tilde{R}(n)$ little is known. This is related to the fact that an ℓ_1-norm nonexpansive map may not have a nonexpansive extension. Even in very low dimensions there are open problems on $\tilde{R}(n)$. For example, we do know that $\{1, 2, 3, 4, 5, 6, 8, 12\} \subset \tilde{R}(3)$, but it is not known if 7, 9, 10 or 11 are in $\tilde{R}(3)$.

To study the set $\tilde{R}(n)$ it is more convenient to express it in terms of so called rigid sets. Let us give the definition of a rigid set at once. A sequence $\{s^i\}_{i\geq 0}$ in $(V, \|\cdot\|)$ is called a *rigid sequence*, if its closure is compact and

$$\|s^{k+l} - s^k\| = \|s^l - s^0\| \quad \text{for each } k, l \in \mathbb{N}.$$

A subset S of $(V, \|\cdot\|)$ is called a *rigid set* if it is the closure of a rigid sequence in $(V, \|\cdot\|)$.

Obviously the set of points of a periodic orbit of a nonexpansive map is a rigid set. On the other hand, for every rigid set S in $(V, \|\cdot\|)$ with cardinality p and rigid sequence $\{s^i\}_i$, we can define a map $g : S \to S$ by

$$g(s^i) = s^{i+1}.$$

This map is an isometry with respect to $\|\cdot\|$, and it has a periodic point of minimal period p. From these observations we conclude that

$$\tilde{R}(n) = \{p \in \mathbb{N} \mid \exists S \subset \mathbb{R}^n \ \ell_1\text{-norm rigid set with } |S| = p\}.$$

The problems on $\tilde{R}(n)$ for low dimensions make it very appealing to use a computer. Therefore the following two questions are of interest. First, can we restrict, without loss of generality, to rigid sets that are contained in $\mathbb{Z}^n$? Second, is it sufficient, in order to decide whether or not p is an element of $\tilde{R}(n)$, to look for all rigid sets in a finite subset of $\mathbb{Z}^n$? In this paper we will show that both questions have a positive answer. So far however, we do not have an upper bound on the size of the finite subset of $\mathbb{Z}^n$, that would allow us an exhaustive search for rigid sets in reasonable time, and hence a computer proof that some p is or is not in $\tilde{R}(n)$.

To end the introduction we remark that similar results hold, if we replace the ℓ_1-norm by the sup-norm, i.e.,

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

In this case we note that a sup-norm nonexpansive map always has a nonexpansive extension to the whole of $\mathbb{R}^n$. Furthermore it is proved by Martus [4] that the cardinality of a sup-norm rigid set in $\mathbb{R}^n$ is bounded by $n!2^n$. Moreover, it is conjectured by Nussbaum [6] that the maximum cardinality of a sup-norm rigid set in $\mathbb{R}^n$ is equal to 2^n. Since the set of vertices of the n-dimensional cube forms a sup-norm rigid set, it suffices to show that 2^n is an upper bound. So far the conjecture is proved for $n = 1, 2$ and 3 (see [3]).

It turns out that our analysis also applies to the sup-norm case, and therefore we can obtain the same results for this norm. We will briefly discuss this in the last section.

2. A LEMMA ON POLYHEDRA

To prove the main results we need a lemma on polyhedra.

Definition 2.1. A set $P \subset \mathbb{R}^n$ is a *polyhedron* if there exist a $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$ such that

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

We call the polyhedron *rational*, if the matrix A and vector b can be chosen with rational entries. A vector $x \in \mathbb{R}^n$ is called an *integral vector* if $x \in \mathbb{Z}^n$.

Each polyhedron consists of several, so called, faces. To define a face we first introduce the notion of a subsystem. A system of linear inequalities $A'x \leq b'$ is a *subsystem* of the system of linear inequalities $Ax \leq b$, if $A'x \leq b'$ can be obtained by deleting some (or possibly none) of the inequalities in $Ax \leq b$.

Definition 2.2. A subset F of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is called a *face* if F is nonempty and $F = \{x \in P \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

A face F of a polyhedron P is *minimal* if it does not contain any other face of P.

A characterization of minimal faces is given by Hoffman and Kruskal [2]. The proof of the following version of their theorem can be found in the book by Schrijver [10, Theorem 8.4].

Theorem 2.1. *Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. A set F is a minimal face of P if and only if $\emptyset \neq F \subseteq P$ and*

$$F = \{x \in \mathbb{R}^n \mid A'x = b'\}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$.

The next lemma will be used in the proof of the main result. Before we state the lemma we define the notion of a homogeneous polyhedron. A polyhedron is called *homogeneous* if for every $y \in P$ and $\lambda \geq 1$ one has $\lambda y \in P$.

Lemma 2.1. *Let P be a rational polyhedron. If P is nonempty and homogeneous, then P contains an integral vector. Moreover there exists an a priori upper bound on*

$$\min\{\|x\|_\infty \mid x \in P \text{ integral}\}.$$

Proof. Suppose P is a nonempty, homogeneous, rational polyhedron in $\mathbb{R}^n$. By definition there exist a rational matrix A and rational vector b such that

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

Since

$$\{x \in \mathbb{R}^n \mid Ax \leq b\} = \{x \in \mathbb{R}^n \mid \mu Ax \leq \mu b\},$$

for every $\mu \geq 1$, we can assume that A has integral entries.

As P is nonempty it contains a minimal face, say F. From Theorem 2.1 it follows that there exists a subsystem $A'x \leq b'$ of $Ax \leq b$ such that

$$F = \{x \in \mathbb{R}^n \mid A'x = b'\}.$$

We may assume that A' has linear independent rows, and hence we can write

$$A' = [A_1 A_2],$$

where A_1 is a nonsingular matrix. The equation $A'x = b'$ has a solution

$$(1) \qquad x_0 = \begin{bmatrix} A_1^{-1} b' \\ 0 \end{bmatrix}.$$

With Cramer's rule we obtain that

$$(2) \qquad (A_1^{-1})_{ij} = \frac{(-1)^{i+j} \det(A_1(i.j))}{\det A_1},$$

where $A_1(i,j)$ is obtained by deleting the i-th row and j-th column in the matrix A_1.

Suppose $\eta \geq 1$ is the smallest number such that $\eta b \in \mathbb{Z}^n$. Let M_1 be the least common multiple of the determinants in absolute value of the nonsingular matrices, that are obtained by deleting rows and columns of A.

From (2) and the assumption that A has integral entries we derive that the matrix $M_1 A_1^{-1}$ has integral entries. Hence, using (1), we see that the vector $\bar{x} = \eta M_1 x_0$ is an integral vector. Moreover, as $x_0 \in P$ and $\eta M_1 \geq 1$ it follows from the fact that P is homogeneous that the vector $\bar{x}$ belongs to P.

Let M_2 be the maximum of the determinants in absolute value of the square matrices, that are obtained by deleting rows and columns of A. Now the following inequalties hold

$$\|\bar{x}\|_\infty \leq \eta M_1 \|x_0\|_\infty \leq \eta n M_1 M_2 \|b\|_\infty.$$

Hence there exists an a priori upper bound on

$$\min\{\|x\|_\infty \mid x \in P \text{ integral}\}. \quad \square$$

In the next section we shall associate with each ℓ_1-norm rigid set S a nonempty, homogeneous, rational polyhedron $P_1(S)$. Before we can start the construction of the polyhedron a final definition is required: for a set Q in $\mathbb{R}$ a matrix A will be called a *Q-matrix* if the entries of A are elements of Q.

3. THE POLYHEDRON FOR ℓ_1-NORM RIGID SETS

Let $S = \{s^0, \dots, s^{p-1}\}$ be an ℓ_1-norm rigid set in $\mathbb{R}^n$ with cardinality p and rigid sequence $\{s^l\}_l$. Define the vector

$$s = (s^0, \dots, s^{p-1}) \in \mathbb{R}^{pn}.$$

For each $i \in \{1, \dots, n\}$ there exists a permutation π_i of $(0, \dots, p-1)$ such that

$$s_i^{\pi_i(0)} \leq s_i^{\pi_i(1)} \leq \dots \leq s_i^{\pi_i(p-1)}, \tag{3}$$

where s_i^j is the i-th coordinate of s^j. The system of linear inequalities, generated by the permutations π_i, can be written as $Ds \leq 0$, where D is a $\{0, \pm 1\}$-matrix. For example consider the rigid set

$$S = \{(\sqrt{2},\ \sqrt{3}),\ (-\sqrt{2}, -\sqrt{3}), (\sqrt{3},\ \sqrt{2}),\ (-\sqrt{3}, -\sqrt{2})\}.$$

If we take $s = (\sqrt{2}, \sqrt{3}, -\sqrt{2}, -\sqrt{3}, \sqrt{3}, \sqrt{2}, -\sqrt{3}, -\sqrt{2})$, then the matrix D is given by

$$D = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since S is a rigid set, we have the equations

$$\|s^{k+l \bmod p} - s^k\|_1 - \|s^l - s^0\|_1 = 0 \quad \text{for } 1 \leq l \leq [p/2] \text{ and } 1 \leq k \leq p-1.$$

Here $[p/2]$ denotes the largest integer $m \leq p/2$. By using the inequalities in (3) we can evaluate in each equation the absolute values. In this way we obtain a system of linear equalities. This system of equalities we shall write as $Cs = 0$, where C is a $\{0, \pm 1, \pm 2\}$-matrix. In the previous example the matrix C is given by

$$C = \begin{bmatrix} -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

Define $\delta > 0$ to be the minimal ℓ_1-norm distance among the elements the S. If we select $\epsilon \in \mathbb{Q}$ such that $0 < \epsilon \leq \delta$, then the following inequalities hold

$$-\|s^k - s^l\|_1 \leq -\epsilon \quad \text{for } 0 \leq k < l \leq p-1.$$

Again by using (3) we can evaluate the absolute values in these inequalities. The resulting system of linear inequalities can be written as $Ex \leq -\epsilon$, where E is a $\{0, \pm 1\}$-matrix. In the previous example $\delta = 2(\sqrt{3} - \sqrt{2}) \geq 1/2$, and hence we can take $\epsilon = 1/2$. The matrix E is given by

$$E = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

Define the polyhedron $P_1(S)$ in $\mathbb{R}^{pn}$ by

$$(4) \qquad P_1(S) = \{x \in \mathbb{R}^{pn} \mid \begin{bmatrix} D \\ C \\ -C \\ E \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{bmatrix}\}.$$

Remark that there is some freedom in the construction of $P_1(S)$. First of all there is the choice of the rigid sequence in S, and the choice of ϵ. Furthermore, if two points in S are equal on the i-th coordinate, then there are two permutations, say π_i and μ_i, such that

$$s_i^{\pi_i(0)} \leq s_i^{\pi_i(1)} \leq \ldots \leq s_i^{\pi_i(p-1)},$$

and

$$s_i^{\mu_i(0)} \leq s_i^{\mu_i(1)} \leq \ldots \leq s_i^{\mu_i(p-1)},$$

So, in the case of equal coordinates there are several possibilities in choosing the matrix D. Our results hold for any possible construction of the polyhedron $P_1(S)$. Therefore we do not make a distinction between the different polyhedra $P_1(S)$, we simply select one.

Lemma 3.1. *The polyhedron $P_1(S)$ as defined in* (4) *has the following properties*

(i) *$P_1(S)$ is a nonempty, homogeneous, rational polyhedron.*

(ii) *Suppose that for $x \in \mathbb{R}^{pn}$ we write $x = (x^0, \ldots, x^{p-1})$, where $x^i \in \mathbb{R}^n$ for $0 \leq i \leq p-1$. If $x \in P_1(S)$, then $S_x = \{x^0, \ldots, x^{p-1}\}$ is an ℓ_1-norm rigid set in $\mathbb{R}^n$ of cardinality p.*

Proof. To see that the polyhedron $P_1(S)$ is nonempty it is sufficient to remark that by construction $s = (s^0, \ldots, s^{p-1})$ belongs to $P_1(S)$. Since ϵ is rational, and the entries of the matrices D, C and E are integral-valued, it follows that $P_1(S)$ is a rational polyhedron. To prove that $P_1(S)$ is homogeneous we assume that $y \in P_1(S)$ and $\lambda \geq 1$. Now the following inequalties hold

$$\begin{bmatrix} D \\ C \\ -C \\ E \end{bmatrix} \lambda y \leq \lambda \begin{bmatrix} D \\ C \\ -C \\ E \end{bmatrix} y \leq \lambda \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{bmatrix}.$$

These inequalities imply that $\lambda y \in P_1(S)$, and therefore the polyhedron $P_1(S)$ is homogeneous.

To prove the second property, we assume that $x \in P_1(S)$, and we write $x =$

$(x^0,\ldots,x^{p-1})$, where $x^i \in \mathbb{R}^n$ for $0 \le i \le p-1$. Further we define the set $S_x = \{x^0,\ldots,x^{p-1}\}$ in $\mathbb{R}^n$.

Since $x \in P_1(S)$ the inequalities $Dx \le 0$ and $Ex \le -\epsilon$ hold. These inequalities imply that

$$-\|x^k - x^l\|_1 \le -\epsilon \quad \text{for } 0 \le k < l \le p-1.$$

Therefore the number of elements in the set S_x is equal to p.

As $x \in P_1(S)$ it follows that $Dx \le 0$ and $Cx = 0$, and hence we obtain that

$$\|x^{k+l \bmod p} - x^k\|_1 - \|x^l - x^0\|_1 = 0 \quad \text{for } 1 \le l \le [p/2] \text{ and } 1 \le k \le p-1.$$

From these equalities we can conclude that S_x is an ℓ_1-norm rigid set in $\mathbb{R}^n$ of cardinality p. □

4. RESTRICTION TO INTEGRAL RIGID SETS

In this section we will answer the two questions raised in the introduction. First, we will prove that we can restrict, without loss of generality, to rigid sets that are contained in $\mathbb{Z}^n$. Subsequently we will show that, in order to decide whether or not p is an element of $\tilde{R}(n)$, it is sufficient to look for all rigid sets in a finite subset of $\mathbb{Z}^n$.

We define the set

$$\tilde{R}_I(n) = \{p \in \mathbb{N} \mid \exists S \subset \mathbb{Z}^n \ \ell_1\text{-norm rigid set with } |S| = p\}.$$

Theorem 4.1. *For each $n \in \mathbb{N}$ one has that $\tilde{R}(n) = \tilde{R}_I(n)$.*

Proof. The inclusion $\tilde{R}_I(n) \subset \tilde{R}(n)$ follows by definition. Therefore it suffices to prove $\tilde{R}(n) \subset \tilde{R}_I(n)$. So, suppose that p is an element of $\tilde{R}(n)$. By definition there exists an ℓ_1-norm rigid set $S = \{s^0,\ldots,s^{p-1}\} \subset \mathbb{R}^n$ of cardinality p, with rigid sequence $\{s^i\}_i$. Let $P_1(S)$ be the polyhedron as constructed in (4). This polyhedron satisfies the first property of Lemma 3.1, and therefore we can apply Lemma 2.1 to obtain an integral vector $x \in P_1(S)$. Now the second property of Lemma 3.1 implies that there exists an integral ℓ_1-norm rigid set in $\mathbb{Z}^n$ of cardinality p. Therefore we conclude that $p \in \tilde{R}_I(n)$, and hence the inclusion $\tilde{R}(n) \subset \tilde{R}_I(n)$ holds. □

From this theorem it follows that in order to know that some p is or is not in $\tilde{R}(n)$, it is sufficient to look for rigid sets in $\mathbb{Z}^n$.

Now we shall answer the second question. Remark that if S in $\mathbb{Z}^n$ is an ℓ_1-norm rigid set, then the ℓ_1-distance between two distinct elements in S is at least 1. This allows us to take $\epsilon = 1$ in the construction of $P_1(S)$. So, for each ℓ_1-norm rigid set S in $\mathbb{Z}^n$, with rigid sequence $\{s^i\}_i$ and cardinality p we have that

$$(s^0,s^1,\ldots,s^{p-1}) \in P_1(D) := \{x \in \mathbb{R}^{pn} \mid \begin{bmatrix} D \\ C \\ -C \\ E \end{bmatrix} x \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}\},$$

where the matrices D, C and E are generated as in the definition of the polyhedron $P_1(S)$. Note that the polyhedron $P_1(D)$ only depends on the matrix D or, equivalently, it depends on the set of permutations π_i. Therefore, given the cardinality p and the dimension n, there are at most $(p!)^n$ distinct polyhedra $P_1(D)$.

From Lemma 2.1, it follows that there exists an a priori upper bound on

$$\min\{\|x\|_\infty \mid x \in P_1(D) \text{ integral}\}$$

for each matrix D. We shall denote this upper bound by M_D.

Thus, a strategy to decide whether or not p belongs to $\tilde{R}(n)$, is to look for all integer rigid sets of cardinality p in the finite set

$$\{x \in \mathbb{Z}^n \mid \|x\|_\infty \leq M_D\}$$

for each matrix D. In each of the $(p!)^n$ possibilities we can use the inequalities on the coordinates, that are induced by the matrix D, to reduce the search. Alternatively we can take the biggest M_D over all matrices D and just look for all integer rigid sets, ignoring the inequalities that come from the matrix D.

So far, however, we do not have an upper bound on the maximum of M_D that would allow us to do an exhaustive search for integral rigid sets.

5. THE ANALOGUE FOR SUP-NORM RIGID SETS

If we replace the ℓ_1-norm by the sup-norm, we can obtain the same results. In fact, the same idea applies. We can associate to each sup-norm rigid set a polyhedron and then prove a lemma similar to Lemma 3.1. We shall outline the construction of the polyhedron.

Suppose that $S = \{s^0, \ldots, s^{p-1}\}$ is a sup-norm rigid set in $\mathbb{R}^n$ of cardinality p. Let $\{s^i\}_i$ be the rigid sequence in S and define the vector

$$s = (s^0, \ldots, s^{p-1}) \in \mathbb{R}^{pn}.$$

For each $i \in \{1, \ldots, n\}$ there exists a permutation π_i of $(0, \ldots, p-1)$ such that

$$s_i^{\pi_i(0)} \leq s_i^{\pi_i(1)} \leq \ldots \leq s_i^{\pi_i(p-1)}, \tag{5}$$

where s_i^j is the i-th coordinate of s^j. We rewrite this set of linear inequalities into a system of linear inequalities $Ds \leq 0$, where D is a $\{0, \pm 1\}$-matrix.

For each pair (k, l) with $0 \leq k < l \leq p-1$ we select an index $i^\star = i_{kl}$ in $\{1, \ldots, n\}$ such that

$$\|s^k - s^l\|_\infty = |s_{i^\star}^k - s_{i^\star}^l|.$$

Now we consider for each pair (k, l) with $0 \leq k < l \leq p-1$ the inequalities

$$|s_i^k - s_i^l| - |s_{i^\star}^k - s_{i^\star}^l| \leq 0 \quad \text{for } i \in \{1, \ldots, n\} \backslash \{i^\star\}. \tag{6}$$

By using (5) we can evaluate the absolute values in each inequality and obtain a system of linear inequalities, which we shall write as $Fs \leq 0$, where F is a $\{0, \pm 1\}$-matrix.

We proceed by looking at the equations

$$\|s^{k+l \bmod p} - s^k\|_\infty - \|s^l - s^0\|_\infty = 0 \quad \text{for } 1 \leq l \leq \lfloor p/2 \rfloor \text{ and } 1 \leq k \leq p-1.$$

In this case we can use (5) and (6) to turn this set of equalities into a system of linear equalities. This system we shall write as $Gs = 0$, where G is a $\{0, \pm 1, \pm 2\}$-matrix.

If we select $\epsilon > 0$ rational such that the minimal sup-norm distance among different elements of S is at least ϵ, then

$$-\|s^k - s^l\|_\infty \leq -\epsilon \quad \text{for } 0 \leq k < l \leq p-1.$$

By using (5) and (6) we can turn this set of inequalities into a system of linear inequalities, say $Hs \leq -\epsilon$, where H is a $\{0, \pm 1\}$-matrix.
Now we define the polyhedron $P_\infty(S)$ in $\mathbb{R}^{pn}$ by

$$P_\infty(S) = \{x \in \mathbb{R}^{pn} \mid \begin{bmatrix} D \\ F \\ G \\ -G \\ H \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\epsilon \end{bmatrix} \}. \tag{7}$$

Again note that there is some freedom in the construction of the polyhedron $P_\infty(S)$. The following lemma, however, will hold for any possible construction of $P_\infty(S)$.

Lemma 5.1. *The polyhedron $P_\infty(S)$ as defined in (7) has the following properties.*

(i) *$P_\infty(S)$ is a nonempty, homogeneous, rational polyhedron.*

(ii) *Suppose that for $x \in \mathbb{R}^{pn}$ we write $x = (x^0, \ldots, x^{p-1})$, where $x^i \in \mathbb{R}^n$ for $0 \leq i \leq p-1$. If $x \in P_\infty(S)$, then $S_x = \{x^0, \ldots, x^{p-1}\}$ is a sup-norm rigid set in $\mathbb{R}^n$ of cardinality p.*

This lemma can be used to prove analogous results for sup-norm rigid sets.

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