Integration

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Let $f(x)$ be a function whose graph is given below. How can one find the area of the shaded region?
Answer: Integration.

Hence, integration can be described as the process of finding the area under a curve.
Area = \int_a^b f(x)\,dx.

- \(a\) is called the lower limit of the integration.
- \(b\) is called the upper limit of the integration.
- \(f(x)\) is called the integrand.
- \(dx\) indicates that the integration is done wrt the variable \(x\).
Another scenario!

Let $y = x^2 + c$, where $c$ is a constant. We know that $\frac{dy}{dx} = 2x$.

Que: How can one recover the function $y$ from $\frac{dy}{dx}$?
Ans: Integration.

Hence, integration can also be described as the process of finding the antiderivative of a function.

Note, \( \int f(x)\,dx \) is the antiderivative of the function \( f(x) \).
What is the difference between $\int_a^b f(x) \, dx$ and $\int f(x) \, dx$?
\[ \int_{a}^{b} f(x) \, dx \]

- Has limits.
- It’s constant when \( a \) and \( b \) are constant.
- Called integration with limits (definite integral).

\[ \int f(x) \, dx \]

- Has no limits.
- It’s a family of functions.
- Called integration without limits (indefinite integral).
Let \( f(x) = ax^n \), then

\[
\int f(x) \, dx = \int ax^n \, dx = a \int x^n \, dx = \frac{a}{n+1}x^{n+1} + c,
\]

where \( a, c \) are constants.
Let \( f(x) = cg(x) \pm h(x) \), then

\[
\int f(x) \, dx = c \int g(x) \, dx \pm \int h(x) \, dx.
\]
Find the integrals of the following functions:

1. $f(x) = 2x$.
2. $f(t) = 3t^2 + t + 1$.
3. $f(x) = \frac{1}{2}x^{-3/4} + \frac{1}{x^2} - \sqrt{x}$. 
Answer:

1. \[ \int 2x \, dx = 2 \int x \, dx = x^2 + c \] because \[ \frac{d}{dx}(x^2 + c) = 2x. \]

Hence, \( 2x \) is the derivative of \( x^2 + c \), and \( x^2 + c \) is the antiderivative of \( 2x \).
Answer:

1. \( \int 2x \, dx = 2 \int x \, dx = x^2 + c \) because \( \frac{d}{dx}(x^2 + c) = 2x \).

   Hence, \( 2x \) is the derivative of \( x^2 + c \), and \( x^2 + c \) is the antiderivative of \( 2x \).

2. \( \int (3t^2 + t + 1) \, dt = t^3 + \frac{1}{2}t^2 + t + c \)
Answer:

1. \[
\int 2x \, dx = 2 \int x \, dx = x^2 + c \text{ because } \frac{d}{dx}(x^2 + c) = 2x.
\]

Hence, \(2x\) is the *derivative* of \(x^2 + c\), and \(x^2 + c\) is the *antiderivative* of \(2x\).

2. \[
\int (3t^2 + t + 1) \, dt = t^3 + \frac{1}{2}t^2 + t + c
\]

3. \[
\int (\frac{1}{2}x^{-3/4} + \frac{1}{x^2} - \sqrt{x}) \, dx = 2x^{1/4} - x^{-1} - \frac{2}{3}x^{3/2} + c.
\]
Example

Calculate:

1. \( \int_0^1 (3t^2 + t + 1) \).
2. \( \int_0^\pi \sin t \, dt \).

Answers

1. \( \int_0^1 (3t^2 + t + 1) = [t^2 + \frac{1}{2}t^2 + t]_0^1 = 2.5 \)
Example

Calculate:

1. $\int_{0}^{1} (3t^2 + t + 1)$.
2. $\int_{\pi}^{x} \sin t \, dt$.

Answers

1. $\int_{0}^{1} (3t^2 + t + 1) = [t^2 + \frac{1}{2}t^2 + t]_{0}^{1} = 2.5$
2. $\int_{\pi}^{x} \sin t \, dt = -[\cos t]_{\pi}^{x} = - \cos x - 1$. 
Calculate

1. \( \int \sqrt{1 + 2t} \, dt \).

2. \( \int_{1-3x}^{1} \frac{u^3}{1 + u^2} \, du \).
Answer:

1. Let \( v = 1 + 2t \) \( \Rightarrow \frac{dv}{dt} = 2 \Rightarrow \frac{dv}{2} = dt \). Hence,

\[
\int \sqrt{1+2t} dt = \frac{1}{2} \int \sqrt{v} dv = \frac{1}{3} v^{3/2} + c = \frac{1}{3} (1 + 2t)^{3/2} + c.
\]

(Integration by substitution)
Answer:

1. Let \( v = 1 + 2t \Rightarrow \frac{dv}{dt} = 2 \Rightarrow \frac{dv}{2} = dt \). Hence,

\[
\int \sqrt{1+2t}dt = \frac{1}{2} \int \sqrt{v}dv = \frac{1}{3}v^{3/2} + c = \frac{1}{3}(1+2t)^{3/2} + c.
\]

(Integration by substitution)

2.

\[
\int_{1-3x}^{1} \frac{u^3}{1+u^2}du = \int_{1-3x}^{1} (u - \frac{u}{1+u^2})du
\]

\[
= \frac{1}{2}[u^2 - \ln(1+u^2)]_{1-3x}^{1}
\]

\[
= \frac{1}{2}(1 - \ln 2) - \frac{1}{2}((1-3x)^2 - \ln(1+(1-3x)^2)).
\]
Application

The $MC$ and $MR$ of a company are given by $2Q - 4$ and $200 - 4Q$ respectively. Find the company’s $TC$ and $TR$ functions. Note, $TR = 0$ and $TC = 100$ when $Q = 0$. 
Introduction

Examples

Application

\[ MC = \frac{dTC}{dQ} = 2Q - 4 \]

\[ dTC = (2Q - 4)dQ \]

\[ \int dTC = \int (2Q - 4)dQ \]

\[ TC = Q^2 - 4Q + c. \]

When \( Q = 0 \), \( TC = 100 \). Hence,

\[ TC = Q^2 - 4Q + 100. \]
\[ MR = \frac{dTR}{dQ} = 200 - 4Q \]

\[ TR = \int (200 - 4Q) dQ = 200Q - 2Q^2 + c \]

When \( Q = 0 \), \( TR = 0 \). Hence,

\[ TR = 200Q - 2Q^2. \]
A particle moves along a line so that its velocity at time $t$ is

$$v(t) = t^2 - t - 6.$$ 

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.

2. Find the distance traveled during the time period.
1. 

\[ s(4) - s(1) = \int_{1}^{4} v(t)dt = \int_{1}^{4} (t^2 - t - 6)dt = -4.5. \]

This means that the particle moved 4.5 m towards left.
2. \( v(t) = t^2 - t - 6 = (t - 3)(t + 2) \) and so \( v(t) \leq 0 \) on the interval \([1, 3]\) and \( v(t) \geq 0 \) on the interval \([3, 4]\).

Total distance

\[
\int_1^4 |v(t)| \, dt = \int_1^3 [-v(t)] \, dt + \int_3^4 v(t) \, dt
\]

\[
= \int_1^3 (t^2 - t - 6) \, dt + \int_3^4 (t^2 - t - 6) \, dt
\]

\[
= \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^3 + \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4
\]

\[
= 61/6 = 10.17m.
\]
Recommended Book:  \textit{Calculus By James Stewart}.
To book a maths/stats appointment...