Implied volatility

Financial Mathematics Clinic

SLAS – University of Kent
1. Introduction

2. Glossary

3. Motivation

4. Implied volatility

5. Possible Solutions
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4. Implied Volatility

5. Possible Solutions
These slides are (mainly) aimed to

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- Postgraduate students.
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Objective

- Understand one of the greatest limitations of the Black-Scholes-Merton model and possible ways to address it.
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5 Possible Solutions
Frictionless market. A market is frictionless when there are no transaction costs, no dividends and no delays in obtaining information or taking decisions.

Convex function. A real-valued function is convex if the line segment between any two points lies above the graph of the function.

At the money (ATM) call (put). ATM are calls and puts whose strike is close to the market price.

In the money (ITM) call (put). A call (put) option is ITM if the market price is above (below) the strike.

Out the money (OTM) call (put). A call (put) option is OTM if the market price is below (above) the strike.
INTRODUCTION

GLOSSARY

MOTIVATION

IMPLIED VOLATILITY

POSSIBLE SOLUTIONS
The Black-Scholes-Merton (BSM) model is one of the most important and influential financial models of all time.
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However, it builds on assumptions that are not perfectly practical (e.g. frictionless market, constant risk-free rate and constant volatility $\sigma$).

There is a need to consider more sophisticated models that overcome these limitations.
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The *smile effect* is an empirical fact that the BSM model fails to explain. In order to analyse it we need to consider:

\[ dS_t = \mu S_t \, dt + \sigma W_t. \]

The rational price of the standard European call option, i.e.

\[ C = C(\sigma, K, T) = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2). \]

The actual market prices, which we define as \( \hat{C}(T, K) \).
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- The *actual* market prices, which we define as \( \hat{C}(T, K) \).
We compare $C(\sigma, T, K)$ with $\hat{C}(T, K)$ and define the *implied volatility* $\hat{\sigma}$ as the solution to

$$f(\sigma) = C(\sigma, T, K) - \hat{C}(T, K)$$

We can use Newton-Rhapson to find the solution.

$$\sigma_n = \sigma_{n-1} - \frac{f(\sigma_{n-1})}{f'(\sigma_{n-1})}$$

In practice $\hat{\sigma}(T, K)$ is not constant.

1. Changes with $T$ for fixed $K$
2. Changes with $K$ for fixed $T$ as a convex function (hence the name *smile effect*). This change appears to be more delicate.
Smile effect (continuation)

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2. Changes with $K$ for fixed $T$ as a convex function (hence the name smile effect). This change appears to be more delicate.
Using Facebook’s daily stock information (available for free at yahoo finance), we can plot the volatility smile of Facebook’s call-options for 5 March 2021. The closing price of the stock was 257.74.
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**Insights**

- BSM model assumes constant volatility which is not true.
- ITM and OTM options have higher volatility.
- The lower volatilities are seen with ATM options.
- Sometimes we see a *smirk* rather an a *smile*. *Reverse skews*, where ITM options have higher volatility and *Forward skews* where OTM options have higher volatility are examples of this.
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Some of the modifications

- Merton proposed considering $\mu = \mu(t)$ and $\sigma = \sigma(t)$. 

Local volatility. We consider $\mu = \mu(t)$ and $\sigma = \sigma(S_t, t)$ (Dupire).

Stochastic volatility. Replace $\sigma$ for $\nu_t$ that models the variance of $S_t$ and consider the model

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} \, dW_t$$

$d\nu_t = \alpha(t, \nu_t) \, dt + \beta(t, \nu_t) \, dB_t$

For example, Heston model:

$$\alpha(t, \nu_t) = \theta(\omega - \nu_t), \quad \beta(t, \nu_t) = \xi \sqrt{\nu_t}$$

GARCH model:

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\begin{align*}
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- GARCH model: $\alpha(t, \nu_t) = \theta(\omega - \nu_t), \beta(t, \nu_t) = \xi \nu_t$
To book a maths/stats appointment...

www.kent.ac.uk/learning
QUESTIONS?