Mean Variance Portfolio Theory

Financial Mathematics Clinic

SLAS – University of Kent

Introduction

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Objective

- Understand the basic principles governing the Mean Variance Portfolio Theory.

Glossary

- *Lagrange multipliers*. A mathematical optimisation method for finding the extrema of a multivariate function subject to equality constraints. For example, for the maximum,

$$\max_{\mathbf{x}} f(\mathbf{x})$$

$$\text{subject to: } g_i(\mathbf{x}) = 0 \qquad\qquad i = 1, ..., M$$

- *Short selling*. It is usually thought as the sale of an asset or stock that the seller does not own. This strategy is used when the investor expects a price decline.

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  - maximising the return given a specified risk.
- One of the basic principles is that we can reduce the risk through diversification.

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- All expected returns, variances and covariances of returns are known.
- We are not considering risk free assets.

Diversification

One of the key principles of MPT is the fact that through diversification we can reduce the risk. If we have n assets, X_i, we build a portfolio as:

$$P = \sum_{i=1}^{n} w_i X_i,$$

where,

- $\sum_{i=1}^{n} w_i = 1$.
- The weights, w_i's, can be negative (*short selling*).

Properties of portfolio P

- Expected return:

$$\mathbb{E}(P) = \sum_{i=1}^{n} w_i \mathbb{E}(X_i)$$

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- Variance:

$$\begin{aligned} \text{Var}(P) &= \sum_{i=1}^{n} w_i^2 \text{Var}(X_i) + 2 \sum_{i<j} w_i w_j \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i<j} w_i w_j \sigma_{ij} \end{aligned}$$

where σ_{ij} is the covariance between assets X_i and X_j.

Some remarks

- The covariance can be written in terms of the correlation of the assets, hence

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  with $\rho_{ij} \in [-1, 1]$.
- If $\rho_{ij} = 1$ (or $\rho_{ij} = -1$) we say the assets are perfectly positively (negatively) correlated and zero standard deviation portfolios can be obtained!

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- The key idea is that for a given return we need to minimise the risk, i.e.,

$$\min \operatorname{Var}(P)$$

subject to:

$$\mathbb{E}(P) = R$$

$$\sum_{i=1}^{n} w_i = 1$$

To solve it we use *Lagrange multipliers*.

FURTHER REMARKS

- Solving for different returns (or simply trying all possible combinations of weights) we can construct the mean-variance frontier (or efficient frontier), which contains the portfolios that:
  - Minimise the risk given a return or
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  - Maximise the return for a given level of risk.
- There exists a portfolio with the minimum variance (minimise the variance without the restriction on the expected return).
- The optimal portfolio will line on the frontier and right to the minimum variance portfolio.

2 Assets Example

Consider assets A and B such that $\mathbb{E}(A) = .5$, $\mathbb{E}(B) = .9$, $\sigma_A = .1$, $\sigma_B = .2$ and $\sigma_{A,B} = .01$ and you want to obtain the mean variance frontier and the minimum variance portfolio.

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- For two assets, the minimum variance portfolio occurs when

$$w_A = \frac{\sigma_B - \sigma_{A,B}}{\sigma_A - 2\sigma_{A,B} + \sigma_B} = .6785714,$$

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$$\mathbb{E}(P_{MV}) = .6285714 \quad \text{and} \quad \text{Var}(P_{MV}) = .07107143$$

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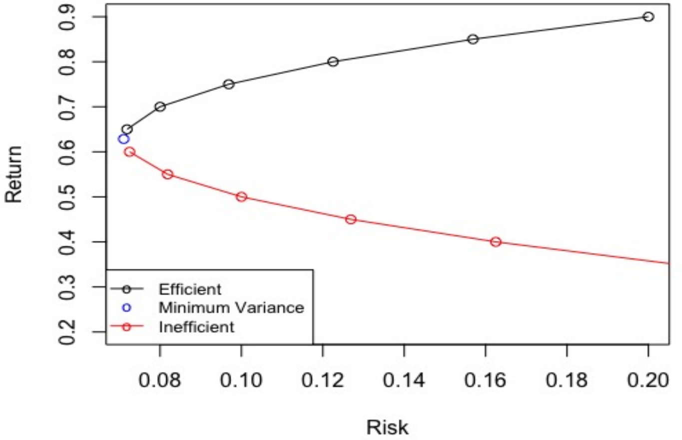
- For risk averse investors the minimum variance portfolio will always be an optimal choice.
- Portfolios with lower return than the P_{MV} will have higher risk (they are inefficient!).
- We can construct the efficient frontier by solving the constrained optimisation problem for returns higher than the one of the P_{MV}.

Example (cont.)



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QUESTIONS?