Mathematical Formulae Sheet

Prepared for SLAS

Differentiation

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, constant	0
x^n, any constant n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\text{cosec } x = \frac{1}{\sin x}$	- $\text{cosec } x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	-$\text{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$-\sinh x$
$\tanh x = \frac{\sinh x}{\cosh x}$	$\text{sech}^2 x$
$\text{cosech} x = \frac{1}{\sinh x}$	-$\text{cosech} x \coth x$
$\text{sech} x = \frac{1}{\cosh x}$	$\text{sech} x \tanh x$
$\coth x = \frac{\cosh x}{\sinh x}$	$-\text{cosech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{-1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1+x^2}$

The linearity rule for differentiation

$$\frac{d}{dx}(au + bv) = a\frac{du}{dx} + b\frac{dv}{dx} \quad a, b \text{ constant}$$

The product and quotient rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The chain rule rule for differentiation
If $y = y(u)$ where $u = u(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Integration

$f(x)$	$\int f(x)dx = F(x) + c$	
k, constant	$kx + c$	
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$	
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$	
e^x	$e^x + c$	
$\cos x$	$\sin x + c$	
$\sin x$	$-\cos x + c$	
$\tan x$	$\ln(\sec x) + c$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$
$\text{cosec} x$	$\ln(\text{cosec} x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln(\cosh x) + c$	
$\coth x$	$\ln(\sinh x) + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$	$\lvert x\rvert > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$	$\lvert x\rvert < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}(\frac{x}{a}) + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}(\frac{x}{a}) + c$	$x \geq a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln\left(x + \sqrt{x^2+k}\right) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(\frac{x}{a}) + c$	$-a \leq x \leq a$
$f(ax+b)$	$\frac{1}{a}F(ax+b) + c$	$a \neq 0$
e.g $\cos(2x-3)$	$\frac{1}{2}\sin(2x-3) + c$	

The linearity rule for integration

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

Integration by substitution

$$\int f(x)\frac{du}{dx}dx = \int f(u)du \text{ and } \int_a^b \frac{du}{dx}dx = \int_{u(a)}^{u(b)} f(u)du$$

Integration by parts

$$\int_a^b v\frac{du}{dx}dx = [uv]_a^b - \int_a^b \frac{dv}{dx}u(x)dx$$

1 Vector Calculus

$$\text{grad} \equiv \nabla \;\; \text{div} \equiv \nabla. \;\; \text{curl} \equiv \nabla\times$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\text{Laplacian} \equiv \nabla^2 \equiv \text{div(grad)} \equiv \nabla.\nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If $\Phi(x,y,z)$ is a scalar field and $\mathbf{v}(x,y,z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is a vector field

$$\text{grad}\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k} \text{ a vector.}$$

$$\text{div}\mathbf{v} = \nabla.\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \text{ a scalar}$$

$$\text{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \text{ a vector}$$

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Vector Calculus identities
$\text{grad}(\Phi\Psi) = \Phi\text{grad}\Psi + \Psi\text{grad}\Phi$
$\text{div}(\Phi a) = \Phi\text{div } a + a\cdot\text{grad}\Phi$
$\text{curl}(\Phi a) = \Phi\text{curl } a + \text{grad}\Phi\times a$
$\text{curlgrad}\Phi = 0, \quad \text{div curl}a = 0$
$\text{curl curl}a = \text{grad div}a - \nabla^2 a$
$\text{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\text{grad})\mathbf{a} + (\mathbf{a}\cdot\text{grad})\mathbf{b} + \mathbf{b}\times\text{curl}\mathbf{a} + a\times\text{curl}\mathbf{b}$
$\text{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\text{curl}\mathbf{a} - \mathbf{a}\cdot\text{curl}b$
$\text{curl}(a\times b) = (\mathbf{b}.\text{grad})\mathbf{a} + a\text{div}\mathbf{b} - \mathbf{b}\text{div}\mathbf{a}$

Green's Theorem in the Plane

$$\oint_C (Pdx + Qdy) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right)$$

Stokes's theorem

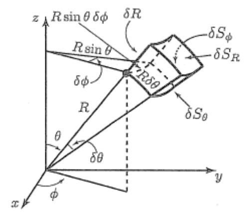
$$\oint_C \mathbf{v}\cdot dr = \int_S \text{curl}\mathbf{v}\cdot d\mathbf{S}.$$

The divergence Theorem

$$\oint_S \mathbf{v}\cdot dS = \int_V \text{div}\mathbf{v}\,dV.$$

Spherical Polar Coordinates

The spherical polar coordinates is a triple (R,θ,ϕ)



$$\begin{cases} x = R\sin\theta\cos\phi & R \geq 0 \\ y = R\sin\theta\cos\phi & 0 \leq \theta\pi \\ z = R\cos\theta & 0 \leq \phi < 2\pi \end{cases}$$

If $\mathbf{v} = v_R\hat{e}_R + v_\theta\hat{e}_\theta + v_\phi\hat{e}_\phi$:

$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{1}{R\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{e}_\phi$$

$$\nabla\cdot v = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 v_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}(v_\phi)$$

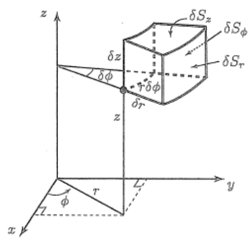
$$\nabla\times v = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{e}_R & R\hat{e}_\theta & R\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_R & Rv_\theta & R\sin\theta v_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Volume element: $\delta V = R^2\sin\theta\,\delta R\,\delta\theta\,\delta\phi$
Surface elements:
$\delta S_R = R^2\sin\theta\,\delta\theta\,\delta\phi$
$\delta S_\theta = R\sin\theta\,\delta R\,\delta\phi$
$\delta S_\phi = R\delta R, \delta\theta$

cylindrical Polar coordinates

The cylindrical coordinates is a triple (r,ϕ,z)



$$\begin{cases} x = r\cos\phi \quad r \geq 0 \\ y = r\sin\phi \quad 0 \leq \phi\pi \\ z = z \end{cases}$$

If $\mathbf{v} = v_r\hat{e}_r + v_\phi\hat{e}_\phi + v_z\hat{e}_z$:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial\Phi}{\partial z}\hat{e}_z$$

$$\nabla\cdot v = \frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(v_\phi) + \frac{\partial v_z}{\partial z}$$

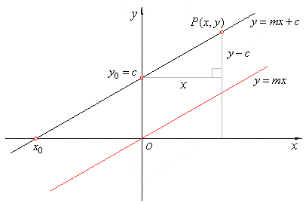
$$\nabla\times v = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ v_r & r\,v_\phi & v_z \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

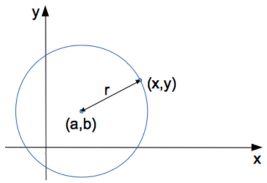
Volume element: $\delta V = r\,\delta r\,\delta\phi\,\delta z$.
Surface elements:
$\delta S_r = r\,\delta\phi\,\delta z$
$\delta S_\phi = \delta r\,\delta z$
$\delta S_z = r\,\delta r\,\delta\phi$

Graph of common of function

Linear: $y = mx + c$, m = gradient c = vertical intercept

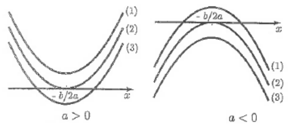


The equation of a circle centre (a,b) with radius r. It is given by



$$(x-a)^2 + (y-b)^2 = r^2$$

Quadratic functions $y = ax^2 + bx + c$



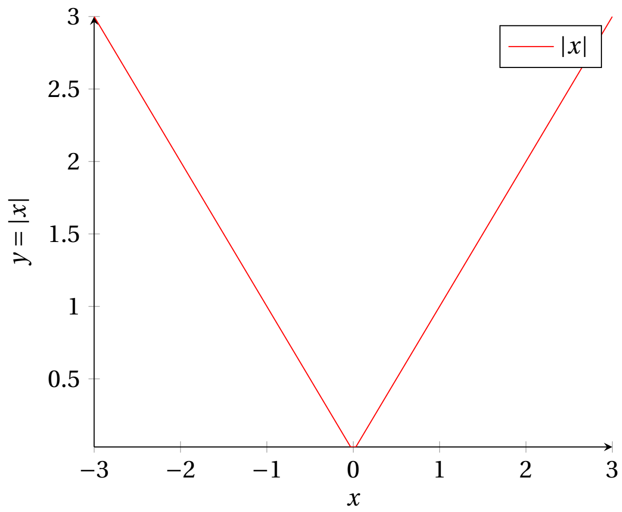
Graph	$a > 0$	$a < 0$
1.	$b^2 - 4ac < 0$	$b^2 - 4ac > 0$
2.	$b^2 - 4ac = 0$	$b^2 - 4ac = 0$
3.	$b^2 - 4ac > 0$	$b^2 - 4ac < 0$

Completing the square If $a \neq 0$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

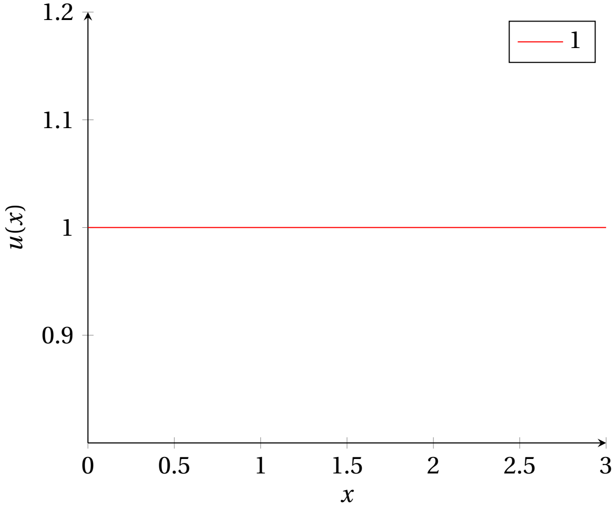
The Modulus function

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$



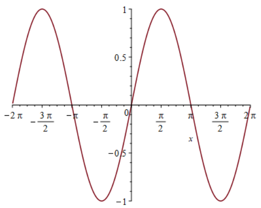
The step function

$$u(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

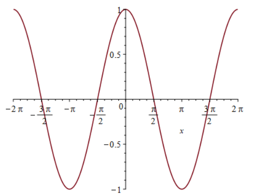


Trigonometry

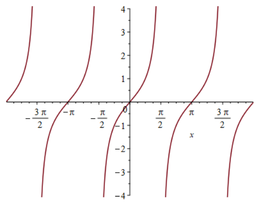
The graph of $\sin x$



The graph of $\cos x$

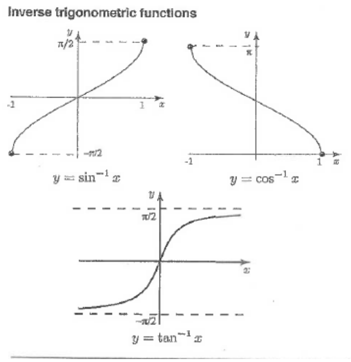


The graph of $\tan x$

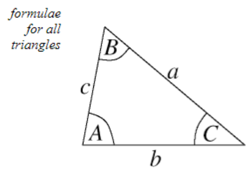


The sine and cosine functions are periodic with period 2π
The tangent is periodic with period π

Inverse of Trigonometric functions

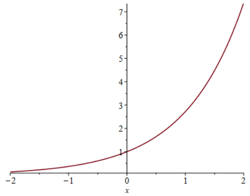


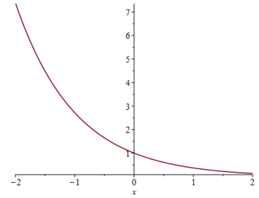
The sine and cosine rule



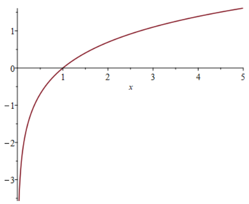
sine rule	cosine rule
$\frac{a}{\sin a} = \frac{b}{\sin b} = \frac{c}{\sin c}$	$a^2 = b^2 + c^2 - 2bc\cos A$

exponential function
The graph of $y = e^x$ and $y = e^{-x}$

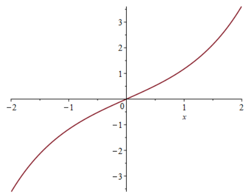




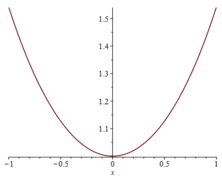
Logarithmic function
Below is the graph of $y = ln(x)$



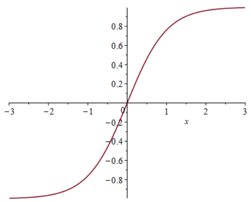
Hyperbolic functions
The graph of $y = \sinh x$



The graph of $y = \cosh x$



The graph of $y = \tanh x$



The Laplace tranform

The **Laplace transform** $f(t)$ is $F(s)$ is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)dt$$

function $f(t), t \geq 0$	Laplace transform $F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$e^a t$	$\frac{1}{s-a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh bt$	$\frac{b}{s^2-b^2}$
$\cosh bt$	$\frac{s}{s^2-b^2}$
$t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$u(t)$ unit step	$\frac{1}{s}$
$\delta(t)$ impulse function	1
$\delta(t-a)$	e^{-sa}
$f(t)$ periodic	$\frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$

Linearity

$$\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}, \;\; \mathscr{L}\{kf\} = k\mathscr{L}\{f\}$$

Shift theorems: If $\mathscr{L}\{f(t)\} = F(s)$ then

$$\mathscr{L}\{e^{-at} f(t)\} = F(s+a).$$

$$\mathscr{L}\{u(t-d)f(t-d)\} = e^{-sd}F(s) \;\; d > 0$$

$u(t)$ is th unit step or Heaviside function

Laplace transform of derivatives and integrals

$$\{f'\} = sF(s) - f(0).$$

$$\mathscr{L}\{f''\}s^2F(s) - sf(s) - f'(0).$$

$$\mathscr{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s).$$

The convolution theorem

The Laplace transform of $f(t) * g(t)$ is $F(s)G(s)$ where

$$f(t) * g(t) = \int_0^t f(t-\lambda)g(\lambda)d\lambda = g(t) * f(t)$$

The z transform

Given a sequence $f[k]$, $k = 0, 1, 3, \ldots$ the $(onesided)$ z transform of $f[k]$ is $F(z)$ defined by

$$F(z) = \mathscr{Z}\{f[k]\} = \sum_{k=0}^{\infty} f[k]z^{-k}$$

sequence $f[k]$	z transform $F(z)$
$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$	1
$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$	$\frac{z}{z-1}$
k	$\frac{z}{(z-1)^2}$
e^{-ak}	$\frac{z}{z-e^{-a}}$
a^k	$\frac{z}{z-a}$
ka^k	$\frac{az}{(z-a)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
$\sin ak$	$\frac{z\sin a}{z^2-2z\cos a+1}$
$\cos ak$	$\frac{z(z-\cos a)}{z^2-2z\cos a+1}$
$e^{-ak}\sin bk$	$\frac{ze^{-a}\sin b}{z^2-2ze^{-a}\cos b+e^{-2a}}$
$e^{-ak}\cos bk$	$\frac{z^2-ze^{-a}\cos b}{z^2-2ze^{-a}\cos b+e^{-2a}}$
$e^{-bk}f[k]$	$F(e^b z)$
$kf[k]$	$-z\frac{d}{dz}F(z)$

Linearity: If $f[k]$ and $g[k]$ are two sequences and c is a constant

$$\mathscr{Z}\{f[k] + g[k]\} = [\mathscr{Z}\{f[k]\}[\mathscr{Z}\{g[k]\}.$$

$$\mathscr{Z}\{cf[k]\}c\mathscr{Z}\{f[k]\}.$$

First shift theorem

$$\mathcal{Z}\{f[k+1]\}zF(z)-zf[0].$$

$$\mathcal{Z}\{f[k+2]\}=z^2F(z)-z^2f[0]-zf[1].$$

Second shift theorem

$$\mathcal{Z}\{f[k-i]u[k-i]\}=z^{-i}F[z],\ i=1,2,3...$$

where $F(z)$ is the z transformof $f[k]$ and $u[k]$ is the unit step sequence

Convolution: $\mathcal{Z}\{f[k]*g[k]\}=F(z)G(z)$ where

$$f[k]*g[k]=\sum_{k=1}^{k}f[m]g[k-m]$$

Numerical Integration

Simpson's rule: For n even and $h\frac{x_n-x_0}{n}$

$$\int_{x_0}^{x_n}f(x)dx\approx\frac{h}{3}(f_0+4f_1+2f_2+4f_3+$$
$$+...+2f_{n-2}+4f_{n-1}+f_n)$$

Truncation error $\approx-\frac{(x_n-x_0)h^d f^{(4)}(\xi)}{180}$

n point Gauss-Legendre formula

$$\int_{-1}^{1}f(x)dx=\sum_{i=1}^{n}w_if(x_i)$$

n	x_i	w_i
2	±0.5777350	1.000000
3	±0.774597	0.555556
	0.0	0.888889
4	±0.861136	0.347855
	±0.339981	0.652145
5	±0.906180	0.236927
	0.0	0.568889
	±0.538469	0.478629

Fourier Series

If $f(t)$ is periodic with period T its Fourier series is given by

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi t}{T}+b_n\sin\frac{2n\pi t}{T}\right)$$

or equivalently, if $\omega=\frac{2\pi}{T}$,

$$f(t)\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega t+b_n\sin(n\omega t)).$$

a_n and b_n are called **Fourier coefficient** givn by

$$a_n=\frac{2}{T}\int_d^{d+T}F(t)\cos\frac{2n\pi t}{T}dt,\text{ for }\ n=0,1,2,..$$

$$b_n=\frac{2}{T}\int_d^{d+T}F(t)\sin\frac{2n\pi t}{T}dt,\text{ for }\ n=1,2,..$$

where d can be chosen to have any value.

If $f(t)$ is odd, $a_n\equiv0$ and $f(t)=\sum_{n=1}^{\infty}b_n\sin n\omega t$.

If $f(t)$ is even, $b_n\equiv0$ and $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\omega t$.

Parseval's theorem:

$$\frac{2}{T}\int_0^T\left(f(t)\right)^2dt=\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right).$$

Complex form:

$$f(t)=\sum_{n=-\infty}^{\infty}c_ne^{\frac{j2n\pi t}{T}},\ c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{-\frac{j2n\pi t}{T}}dt$$

Half-range sine series: Given $f(t)$ for $0<t<\frac{T}{2}$, its odd periodic extension has period T and Fourier series given by

$$f(t)=\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi t}{T}.$$

$$b_n=\frac{4}{T}\int_0^{\frac{T}{2}}f(t)\sin\frac{2n\pi t}{T}dt\ \text{ for }\ n=1,2,3,..$$

Half-range cosine series: Given $f(t)$ for $0<t<\frac{T}{2}$, its even periodic extension has period T and Fourier series given by

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi t}{T}.$$

$$a_n=\frac{4}{T}\int_0^{\frac{T}{2}}f(t)\cos\frac{2n\pi t}{T}dt\ \text{ for }\ n=0,1,2,3,..$$

The Fourier transform

The **Fourier transform** of $f(t)$ is $F(\omega)$ defined by

$$\mathcal{F}\{f(t)\}=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt=F(\omega).$$

The **inverse Fourier transform** is given by

$$\mathcal{F}^{-1}\{F(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega=f(t).$$

function $f(t)$	Fourier transform $F(\omega)$
$Au(t)e^{-\alpha t},\ \alpha>0$	$\frac{A}{\alpha+j\omega}$
$\begin{cases}1 & \alpha\le t\ge\alpha\\ 0 & \text{otherwise}\end{cases}$	$\frac{2\sin\omega\alpha}{\omega}$
A constant	$2\pi A\delta(\omega)$
$u(t)A$	$A\left(\pi\delta(\omega)-\frac{j}{\omega}\right)$
$\delta(t)$	1
$\delta(t-a)$	$e^{-j\omega\alpha}$
$\cos at$	$\pi(\delta(\omega+a)+\delta(\omega-a))$
$\sin at$	$\frac{\pi}{j}(\delta(\omega-a)-\delta(\omega+a))$
$\text{sgn}(t)$	$-j\pi\text{sgn}(\omega)$
$e^{-\alpha\|t\|},\ \alpha>0$	$\frac{2\alpha}{\alpha^2+\omega^2}$

Linearity

$$\mathscr{F}\{f+g\}=\mathscr{F}\{f\}+\mathscr{F}\{g\},\ \ \mathscr{F}\{kf\}=k\mathscr{F}\{f\}.$$

Shift theorems: If $F(\omega)$ is the Fourier transform of $f(t)$

$$\mathscr{F}\{e^{jat}f(t)\}=F(\omega-a),\ \ a\text{ constant.}$$

$$\mathscr{F}\{f(t-\alpha)\}=e^{-j\alpha\omega}F(\omega),\ \ \alpha\text{ constant.}$$

Differentiation: The Fourier transform of the nth derivative, $f^{(n)}(t)$, is $(j\omega)^nF(\omega)$.

Duality: If $F(\omega)$ is the Fourier transform of $f(t)$ then the Fourier transform of $F(t)=2\pi\times f(-\omega)$.

Convolution and correlation:

The Fourier transform of $f(t)*g(t)$ is $F(\omega)G(\omega)$ where

$$f(t)*g(t)=\int_{-\infty}^{\infty}f(\lambda)g(t-\lambda)d\lambda=g(t)*f(t).$$

The Fourier transform of $f(t)*g(t)$ is $F(\omega)G(-\omega)$ where

$$f(t)*g(t)=\int_{-\infty}^{\infty}f(\lambda)g(t-\lambda)d\lambda.$$

Discrete Fourier transform dft

Given a sequence of N terms

$$\{g[0],g[1],g[2],...,g[N-1]\}$$

its discrete Fourier transform (dft) is the sequence

$$\{G[0],G[1],G[2],...,G[N-1]\}$$

where

$$G[k]=\sum_{n=0}^{N-1}G[k]e^{\frac{-2jnk\pi}{N}}.$$

Furthermore

$$g[n]=\frac{1}{N}\sum_{K=0}^{N-1}G[k]e^{\frac{2jnk\pi}{N}}.$$

Maclaurin and Tylor Series

Maclaurin Series:

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^r}{r!}f^{(r)}(0)+...$$

Tylor series (one variable):

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+ ...+\frac{(x-a)^r}{r!}f^{(r)}(a)+...$$

Taylor series (two variables); For a function $f(x,y)$ of two variables

$$f(x,y)=f(a,b)+\frac{1}{1!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)f(a,b)$$
$$+\frac{1}{2!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^2f(a,b)+...$$
$$+\frac{1}{r!}\left((x-a)\frac{\partial}{\partial x}+(y-b)\frac{\partial}{\partial y}\right)^rf(a,b)+...$$

Stationary points in two variables: For $z=f(x,y)$, stationary points are located by solving $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. Define $\Delta=\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$ at (a,b).

$$\Delta<0\ \text{ saddle point}$$

$$\Delta>0\text{ and }\frac{\partial^2 f}{\partial x^2}>0\ \text{ minimum pont}$$

$$\Delta>0\text{ and }\frac{\partial^2 f}{\partial x^2}<0\ \text{ maximum pont}$$

Ordinary Differential Equations

To solve $\frac{dy}{dx}=f(x,y)$

Euler Method:

$$y_{r+1}=y_r+hf(x_r,y_r)$$

Modified Euler method

$$y^{(p)}_{r+1}=y_r+hf_r\ f^{(p)}_{r+1}=f(x_{r+1},y^{(p)}_{r+1}).$$

$$y^{(c)}_{r+1}=y_r+\frac{h}{2}(f_r+f^{(}_{r+1}p)).$$

Runge-Kuta method:

$$k_1=hf(x_r,y_r),\ k_2=hf(x_r+\frac{h}{2},y_r+\frac{k_1}{2})$$

$$k_3=hf(x_r+\frac{h}{2},y_r+\frac{k_2}{2}),\ k_4=hf(x_r+h,y_r+k_3).$$

$$y_{r+1}=y_r+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

Complex numbers

Cartesian form:

$$z=a+bj\ \text{ where } j=\sqrt{-1}$$

Polar form

$$z=r(\cos\theta+j\sin\theta)=r\measuredangle\theta$$

$$a=r\cos\theta,\ b=r\sin\theta,\ \tan\theta=\frac{b}{a}$$

Exponential form: $z=re^{j\theta}$

Multiplication and division in polar form

$$z_1z_2=r_1r_2\angle(\theta_1+\theta_2),\quad \frac{z_1}{z_2}=\frac{r_1}{r_2}\angle(\theta_1-\theta_2$$

If $z=r\angle\theta$ then $z^n=r^n\angle(n\theta)$

De Moivre's theorem

$$(\cos\theta+j\sin\theta)^n=\cos n\theta+j\sin n\theta$$

Relationship between hyperbolic and trig functions
$\cos jx = \cosh x$, $\sin jx = j\sinh x$
$\cosh jx = \cos x$, $\sinh jx = j\sin x$ i rather than j maybe used to denote $\sqrt{-1}$
Vectors if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$
Scalar product

$$a \cdot \mathbf{b} = |a||\mathbf{b}|\cos\theta$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{e}}$$

$\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing **a** and **b** in a sense defined by the right hand screw rule.
If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

sequences and series

Arithmetic progression: $a, a+d, a+2d, \ldots$
a = first term, d = common difference,
k^{th} term $= a + (k-1)d$
Sum of n terms, $S_n = \frac{n}{2}(2a + (n-1)d)$
Sum of the first n integers,
$1 + 2 + 3 + \ldots + n =$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Sum of the squares og the first n integers, $1^2 + 2^2 + 3^2 + \ldots + n^2 =$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}(n+1)(2n+1)$$

Geometric progression: $a, ar, ar^2, \ldots$
a = first term, d = common ratio,
k^{th} term $= ar^{k-1}$
Sum of n terms, $S_n = \frac{a(1-r^n)}{1-r}$ provided $r \neq 1$
Sum of Infinity geometric series:

$$S_\infty = \frac{a}{1-r} \quad -1 < r < 1$$

The binomial theorem
If n is a positive integer

$$(1 + x^n) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1(n-2))}{3!}x^3 + \ldots + x^n$$

When n is a negative or fractional the series is infinite and converges when $-1 < x < 1$
Standard power series expansions

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{1}{2}x^2 + \ldots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$

These holds for all values of x.

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

Only for $-1 < x \leq 1$
The exponential function as the limit of a sequence

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

Matrices and Determinants

The 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The 3×3 matrix

$$A = \begin{pmatrix} a_{11}a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

has determinant

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

expanded along the first row
The inverse of a 2×2 **matrix**
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -ca & \end{pmatrix}$$

provided that $ad - bc \neq 0$
Matrix Multiplication: For 2×2 matrices we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \end{pmatrix}$$

Remember $AB \neq BA$ except in special cases.

The Binomial Coefficients

The coeffiecient of x^k in the binomial expansion of $(1+x)^n$ when n is a positive integer is denoted by $\binom{n}{k}$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Algebra

$$(x+k)(x-k) = x^2 - k^2$$

$$(x+k)^2 = x^2 + 2kx + k^2,\ (x-k)^2 = x^2 - 2kx + k^2$$

Formula for solving quadratic equation:
If $ax^2 + bx + c$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laws of Indices

$$a^m a^n = a^{m+n} \ \frac{a^m}{a^n} = a^{m-n} \ (a^m)^n = a^{mn}$$

$$a^0 = 1 \ a^{-m} = \frac{1}{a^m} \ a^{\frac{1}{n}} = \sqrt[n]{a} \ a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

Law of Logarithms: For any positive number base b with $b \neq 1$

$$\log_b A = c \text{ means } A = b^c$$

$$\log_b A + \log_b B = \log_b AB,\ \log_b A - \log_b B = \log_b \frac{A}{B},$$

$$n \log_b A = \log_b A^n,\ \log_b 1 = 0,\ \log_b b = 1$$

Formula for change of base:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Logarithms to base e, denoted $\log_e$ or alternatively ln are called natural logarithm. The letter e stands for the exponential constant which is approximately 2.718

Partial Fractions
For proper fractions $\frac{P(x)}{Q(x)}$ where P and Q are are polynomials with the degree of P less than the degree of Q:
A linear factor $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$.
Repeated linear fact $(ax+b)^2$ in the denominator produce partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
A quadratic factor $ax^2 + bx + c$ in the denominator produces a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.
Improper fractions require an additional term which is a polynomial of degree $n-d$ where n is the degree of the numerator and d is the degree of the denominator.

Inequalities: 4

$a > b$ means a is greater than b
$a < b$ means a is less that b
$a \geq b$ means a is greater than or equal b
$a \leq b$ means a is less that or equal b

Trigonometry

Degrees and radians

$$360^\circ = 2\pi \text{ radians } 1^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$

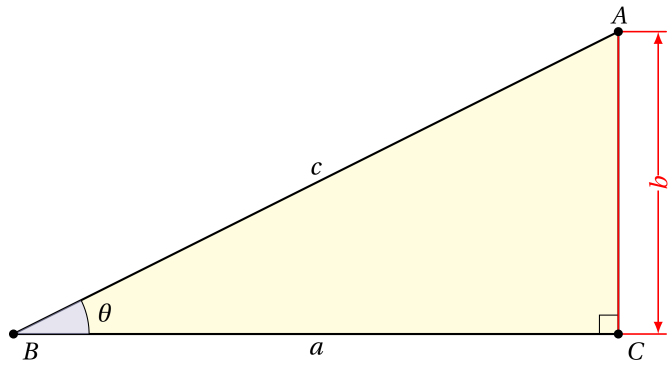
$$1 \text{radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ$$

Trig ratios for an acute angle θ

$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{a}{c}$$

$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{b}{a}$$



Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Standard triangles

$$\sin 45^\circ = \frac{1}{\sqrt{2}},\ \cos 45^\circ \frac{1}{\sqrt{2}},\ \tan 45^\circ = 1$$

$$\sin 30^\circ = \frac{1}{2},\ \cos 30^\circ \frac{\sqrt{3}}{2},\ \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2},\ \cos 60^\circ \frac{1}{2},\ \tan 60^\circ = \sqrt{3}$$

Common trigonometric identities
$\sin(A \pm B) = \sin A cos B \pm \cos A \sin B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\tan(A+B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$2\sin A\cos B = \sin(A+B) + \sim(A-B)$
$2\cos A\cos B = \cos(A-B) + \cos(A+B)$
$2\sin A\sin B = \cos(A-B) - \cos(A+B)$
$\sin^2 A + \cos^2 A = 1$
$1+\cot^2 A = \text{cosec}^2 A,\ \tan^2 A + 1 \sec^2 A$
$\cos 2A == \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
$\sin 2A = 2\sin A\cos A$
$\sin^2 A = \frac{1-\cos 2A}{2},\ \cos A = \frac{1+\cos 2A}{2}$

---

the notation $\sin^2 A$ is used for $(\sin A)^2$ and this notation is consistently used for all other trigonometric functions as well.

2

$$\cosh x = \frac{e^x + e^{-x}}{2},\ \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic Identities

$e^x = \cosh x + \sinh x,\ e^{-x} = \cosh x - \sinh x,\ \cosh^2 x - \sinh^2 x = 1$

$$1 - \tanh^2 x = \text{sech}^2 x$$
$$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh x$$
$$\cosh x \pm y = \cosh x\cosh y \pm \sinh x\sinh y$$
$$\sinh 2x = 2\sinh x\cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Inverse hyperbolic functions

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \text{ for } x \geq 1$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ for } -1 < x < 1$$

The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Functions of A complex Variable

Derivative: If $w = f(z)$ where z and w are complex numbers, the derivative $\frac{dw}{dz}$ at z_0 is

$$f'(z_0) = \lim_{z\to z_0}\left[\frac{f(z) - f(z_0)}{z - z_0}\right]$$

provided that the limit exists at $z \to z_0$ along any path. If $f(z)$ has a derivative at a point z_0 and at all points in some neighborhood of z_0 then $f(z)$ is said to be **analytic** in $\mathbb{R}$.

Cauchy-Riemann equations: If $z = x + jy$ and $w = f(z) = u(x,y) + jv(x,y)$ where x, y, u and v are real variables, and $f(z)$ is analytic in some region R of the z plane, then the **Cauchy-Riemann equations** hold throughout $\mathbb{R}$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If these partial derivatives are continuous in $\mathbb{R}$, the Cauchy-Riemann equations are sufficient conditions to ensure that $f(z)$ is analytic. Furthermore,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

.

Singularities: If $f(z)$ fails to be analytic at point z_0 but is analytic at some point in the neighborhood of z_0 then z_0 is called a singular point of $f(z)$.

Laurent Series: If $f(z)$ is analytic on concentric circles C_1 and C_2 of radii r_1 and r_2, centered at z_0, and also analytic throughout the annular region between the circles, then for each point z within the annulus, $f(z)$ may be represented by the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

in which c_n are complex constants. The series may be written

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z - z_0)^n + \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

. **Poles:** The first sum on the right is the **principal** part. If there are only a finite number of terms in the principal part e.g.

$$f(z) = \frac{c_{-m}}{(z - z_0)^m} + \cdots + \frac{c_{-1}}{(z - z_0)}$$
$$+ c_0 + c_1 (z - z_0) + \cdots + c_m (z - z_0)^m + \cdots$$

in which $c_{-m} \neq 0$, then $f(z)$ has a singularity called a **pole of order** m at $z = z_0$. A pole of order 1 is called a

simple pole. If there are infinitely many terms in the principal part, z_0 is called an **isolated singularity**. If the principal part is zero, then $f(z)$ has a **removable singularity** at z=z_0 and the Laurent series reduces to a Taylor series. **Residues:** If $f(z)$ has a pole a $z = z_0$ then the coefficient, c_1, of $\frac{1}{z-z_0}$ in the Laurent expansion is called the **residues** of $f(z)$ at $z = z_0$. The residue at a pole of order m is given by:

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}$$

When evaluating the integrals that follow, the curve C is traversed in an anticlockwise sense. **Cauchy's theorem:** If $f(z)$ is analytic within and on a simple closed curve C then $\oint_C f(z)dz = 0$. **Cauchy's integral formula:** If $f(z)$ is analytic within and on a simple closed curve C, and if z_0 is any point within C then

$$\oint_C f(z)dz = 2\pi j f(z_0)$$

. Further

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi j}{n!} f^{(n)}(z_0)$$

The residue theorem: If $f(z)$ is analytic within and on a simple closed curve C apart from a finite number of poles inside C, then

$$\oint_C f(z)dz =$$

$2\pi j \times$ [Sum of residues of $f(z)$ at the poles inside C]