Implied volatility

Financial Mathematics Clinic

SLAS – University of Kent

Introduction

These slides are (mainly) aimed to

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- Postgraduate students.

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Objective

- Understand one of the greatest limitations of the Black-Scholes-Merton model and possible ways to address it.

Glossary

- *Frictionless market.* A market is frictionless when there are no transactions costs, no dividends and no delays in obtaining information or taking decisions.
- *Convex function.* A real-valued function is convex if the line segment between any two points lies above the graph of the function.
- *At the money (ATM) call (put).* ATM are calls and puts whose strike is close to the market price.
- *In the money (ITM) call (put).* A call (put) option is ITM if the market price is above (below) the strike.
- *Out the money (OTM) call (put).* A call (put) option is OTM if the market price is below (above) the strike.

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- It is widely consider to be an equation that changed the world.
- However, it builds on assumptions that are not perfectly practical (e.g. frictionless market, constant risk-free rate and constant volatility σ).
- There is a need to consider more sophisticated models that overcome these limitations.

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- The *actual* market prices, which we define as $\hat{C}(T, K)$.

- We compare $C(\sigma, T, K)$ with $\hat{C}(T, K)$ and define the *implied volatility* $\hat{\sigma}$ as the solution to

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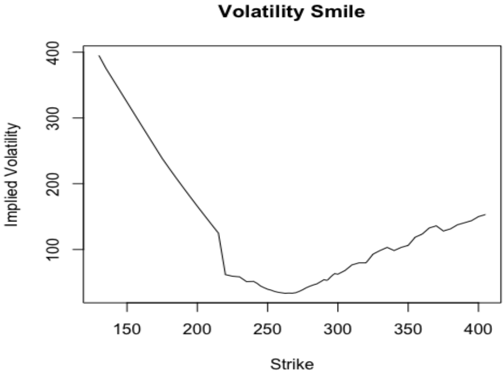
- In practice $\hat{\sigma} = \sigma(T, K)$ is **not constant**.
  1. Changes with T for fixed K and
  2. Changes with K for fixed T as a convex function (hence the name *smile effect*). This change appears to be more delicate.

Real-data example

Using Facebook's daily stock information (available for free at yahoo finance), we can plot the *volatility smile* of Facebook's call-options for 5 March 2021. The closing price of the stock was 257.74.



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- ITM and OTM options have higher volatility.
- The lower volatilities are seen with ATM options.
- Sometimes we see a *smirk* rather an a *smile*. *Reverse skews*, where ITM options have higher volatility and *Forward skews* where OTM options have higher volatility are examples of this.

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- *Stochastic volatility*. Replace σ for ν_t that models the variance of S_t and consider the model

$$\begin{aligned} dS_t &= \mu S_t d_t + \sqrt{\nu_t} dW_t \\ d\nu_t &= \alpha(t, \nu_t)dt + \beta(t, \nu_t)dB_t \end{aligned}$$

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- GARCH model: $\alpha(t, \nu_t) = \theta(\omega - \nu_t)$, $\beta(t, \nu_t) = \xi \nu_t$

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QUESTIONS?