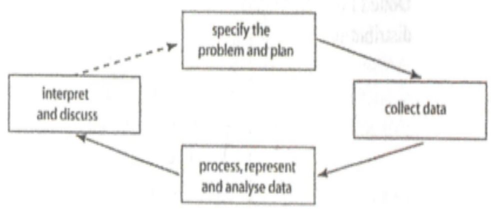
Statistics Notebook

September 26, 2023

The statistics problem solving cycle

Data are numbers in context and the goal of statistics is to get information from those data usually through *problem solving*. A procedure or paradigm for statistical problem solving and scientific enquiry is illustrated in the diagram. The dotted line means that, following discussion, the problem may need to be re-formulated and at least more than one iteration completed.



Descriptive Statistics

Given a sample of n observations $x_1, x_2, ..., x_n$ we define the **sample mean** to be

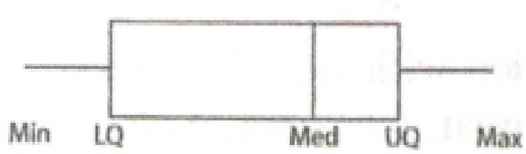
$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

and the *corrected* sum of squares (measures the total variability in the sample from the mean) by

$$S_{xx} = \sum (x_i - \bar{x}) = \sum x_i^2 - n\bar{x} = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$\frac{S_{xx}}{n}$ is sometimes called the *mean square deviation*. An **unbiased estimator** of the population variance, σ^2, is $s^2 = \frac{S_{xx}}{(n-1)}$. The **sample standard deviation** is s. In calculating s^2, the divisor $(n-1)$ is called the **degrees of freedom (df)**. Note that s is also sometimes written $\hat{\sigma}$.

If the sample mean data are ordered from the smallest to largest then the:
minimum (Min) is the smallest value;
lower quartile (LQ) is the $\frac{1}{4}(n+1)$-th value;
median (Med) is the middle [or the $\frac{1}{2}(n+1)$-th] value;
upper quartile (UQ) is the $\frac{3}{4}(n+1)$-th value;
maximum (Max) is the largest value.
These five values constitute a **five-number summary** of the data. They can be represented diagrammatically by a *box-and-whisker plot*, commonly called *boxplot*.



Grouped Frequency Data

If the data are given in the form of a grouped frequency distribution where we have f_i observations in an interval whose mid-point is x_i then, if $\sum f_i = n$

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{\sum f_i x_i}{n}$$

$$S_{xx} = \sum f_i(x_i - \bar{x})^2 = \sum f_i x_i^2 - \frac{(\sum f_i x_i)^2}{n}$$

Events & probabilities

The *intersection* of two events A and B is $A \cap B$. The union of A and B is $A \cup B$. A and B are **mutually exclusive** if they cannot both occur, denoted $A \cap B = \emptyset$, where $\emptyset$ is called the **null event**. For an event A, $0 \leq P(A) \leq 1$. For two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive then,

$$P(A \cup B) = P(A) + P(B)$$

Equally likely outcomes If a complete set of n elementary outcomes are all equally likely to occur, then the probability of each elementary outcomes is $\frac{1}{n}$. If an event A consists of m of these n elements then $P(A) = \frac{m}{n}$.

Independent events A, B are *independent* if and only if $P(A \cap B) = P(A)P(B)$.

Conditional Probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad if \quad P(B) \neq 0$$

Bayes' Theorem: $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

Theorem of Total Probability The k events $B_1, B_2, ..., B_k$ form a partition of the sample space S, if $B_1 \cup B_2 \cup ... \cup B_k = S$ and no two of the B_i's can occur together. Then $P(A) = \sum P(A|B_i)P(B_i)$. In this case Bayes' Theorem generalizes to

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}, \quad i = 1, 2, ..., k$$

If B' is the *complement* of the event B, $P(B') = 1 - P(B)$ and $P(A) = P(A|B)P(B) + P(A|B')P(B')$ is a special case of the theorem of the total probability. The complement of B is commonly denoted $\bar{B}$.

A **hypothesis test** involves testing a claim, or **null hypothesis** H_0, about a parameter against an alternative, H_1. A decision to **reject** H_0, or **not reject** H_0 uses sample evidence to *calculate* a **test statistic** which is judged again a **critical value**. H_0 is maintained unless it is made untenable by sample evidence. Rejecting H_0, when we should not is a **Type I error**. The probability (we are prepared to accept) of making a Type I error is called **significance level** a and yields the critical value. The *smallest* a at which we can just reject H_0 is the **p-value** of the test. Not rejecting H_0 when we should is a **Type II error**, with probability b. The power of a hypothesis test is $1 - b$. An **interval estimate** for a parameter is a *calculated* range within which it is deemed likely to fall. Given a, the set of intervals from infinitely repeated random samples of size n will contain the parameter $(100 - a)\%$ of the time: each interval is $(100 - a)\%$ **confidence interval**.

One sample Hypothesis Testing

1. For $X \sim N(\mu, \sigma^2)$, σ^2 *known; random sample evidence* $\bar{x}$ *and* n. Null hypothesis, $H_0 : \mu = \mu_0$; 2-sided alternative $H_1 : \mu \neq \mu_0$. Test statistic $z_{calc} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$. Reject the H_0 (at the a level) if $|z_{calc}| \geq z_{a/2}$, the critical value of z.

2. For $X \sim N(\mu, \sigma^2)$, σ^2 *unknown; random sample evidence* $\bar{x}, s$ *and* n. Null hypothesis, $H_0 : \mu = \mu_0$; 2-sided alternative $H_1 : \mu \neq \mu_0$. Test statistic $t_{calc} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{(n-1)}$, the t-distribution with $(n - 1)$ degrees of freedom. For $n > 30$ and if X has any distribution, $t \sim N(0, 1)$. Reject H_0 if $|t_{calc}| \geq t_{a/2}$ the critical value of t with $(n - 1)$ df.

3. For $X \sim N(\mu, \sigma^2), \sigma^2$ *unknown*; *random sample evidence* s *and* n. Null hypothesis $H_0 : \sigma^2 = \sigma_0^2$; alternative $H_1 : \sigma^2 > \sigma_0^2$. Test statistic $x_{calc}^2 = (n - 1)s^2/\sigma_0^2 \sim x_{n-1}^2$. Reject H_0 if $x_{calc}^2 > x_a^2$, the critical value of x^2 with $(n - 1)$ df. In each case the p-value is the tail area outside the calculated statistic.

Two sample hypothesis test

For $X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2), \sigma_1^2, \sigma_2^2$ *unknown; random sample evidence* $\bar{x}_1, \bar{x}_2, s_1^2, s_2^2, n_1$ *and* n_2.

1. Null hypothesis , $H_0 : \mu_1 - \mu_2 = c$; 2-sided alternative $H_1 : \mu_1 - \mu_2 \neq c$. Test statistic $t_{calc} = \frac{(\bar{x_1} - \bar{x_2} - c)}{s\sqrt{1/n_1 + 1/n_2}} \sim t_{(n_1+n_2-2)}$ and $s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{(n_1+n_2-2)}$, assuming $\sigma_1^2 = \sigma_2^2$. Reject H_0 if $|t_{calc}| \geq t_{a/2}$, the critical value of t with $(n_1 + n_2 - 2)$ df.

2. Null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$; alternative $H_1 : \sigma_1^2 > \sigma_2^2$. Test statistic $F_{calc} = \frac{(n_1-1)s_1^2}{(n_2-1)s_2^2} \sim F_{n_1-1, n_2-1}$. Reject H_0 if $F_{calc} > F_a$ the critical value of F with $n_1 - 1$ and $n_2 - 1$ df.

Confidence interval for a population mean- σ^2 unknown

If X has mean μ and variance σ^2, with $n > 30$ an approximate $(100 - a)\%$ confidence interval for μ is $\bar{x} - \frac{t_{a/2}s}{\sqrt{n}}$ to $\bar{x} + \frac{t_{a/2}s}{\sqrt{n}}$. If $X \sim N(\mu, \sigma^2)$ the interval is exact for all n.

Standard statistical distributions

Name/parameters	Conditions/application	pdf/pmf	Mean	Variance	mgf	Notes
Binomial $\text{Bin}(n,p)$ Positive integer n Probability p, $0 \le p \le 1$	n independent success/fail trials each with probability p of success. X = number of successes.	$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,\ldots,n$	np	$np(1-p)$	$(1-p+pe^t)^n$	$X \sim \text{Bin}(n,p)$ $\Rightarrow n-X \sim \text{Bin}(n,1-p)$
Geometric $\text{Geom}(p)$ Probability p, $0 \le p \le 1$	Repeated independent success/fail trials each with probability p of success. X = number of trials up to and including the first success.	$P(X=x)=(1-p)^{x-1}p$ $x=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	Has the "lack of memory" property $P(X>a+b \mid X>b)=P(X>a)$
Poisson $\text{Po}(\lambda)$ λ a positive number	Events occur randomly at a constant rate. X = number of occurrences in some interval. λ is the expected number of occurrences	$P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!}$ $x=0,1,2,\ldots$	λ	λ	$\exp(\lambda(e^t-1))$	Useful as approximation to $\text{Bin}(n,p)$ if n is large and p is small
Normal $N(\mu,\sigma^2)$ μ, σ both real; $\sigma>0$	A widely used distribution for symmetrically distributed random variables with mean μ and standard deviation σ.	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ all real x	μ	σ^2	$\exp\left(\mu t+\frac{1}{2}\sigma^2t^2\right)$	Can approximate Binomial, Poisson Pascal and Gamma distributions (see Central Limit Theorem)
Exponential $\text{Expon}(\theta)$	Events are occurring at rate θ per unit time. X = time to first occurrence.	$f(x)=\theta\exp(-\theta x)$ $x>0$	$\frac{1}{\theta}$	$\frac{1}{\theta^2}$	$\frac{\theta}{\theta-t}$, $t<\theta$	Has the "lack of memory" property $P(X>a+b \mid X>b)=P(X>a)$
Negative-binomial or Pascal $\text{Pasc}(r,p)$ Positive integer n Probability p, $0 \le p \le 1$	Repeated independent success/fail trials each with probability p of success. X = number of trials up to and including the r-th success.	$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x=r,r+1,r+2,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$	$\text{Pasc}(1,p) \equiv \text{Geom}(p)$
Gamma $\text{Ga}(\alpha,\beta)$ $\alpha,\beta>0$	Generalization of the exponential distribution; if α is an integer it represents the waiting time to the α-th occurrence of a random event where β is the expected number of events.	$f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ $x>0$	$\frac{\alpha}{\beta}$ $\alpha>1$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$, $t<\beta$	$\text{Ga}(1,\lambda) \equiv \text{Expon}(\lambda)$ If ν is an integer, $\text{Ga}(\nu/2,2)$ is χ^2_ν, the Chi-squared distribution with ν df.

Statistics & Sampling Distributions Populations and samples

A (statistical) **population** is the complete set of all possible measurements or values, corresponding to the entire collection of units, for which inferences are to be made from taking a **sample-** the set of measurements or values that are actually collected from a population.
Simple random sample: every item in the population is equally likely to be in the sample , independently of which other members of the population are chosen.
Parameter: a quantity that describes an aspect of the population, eg. the population mean μ of variance σ^2.
Statistic: a quantity calculated from the sample, eg. the sample mean $\bar{x}$, or variance s^2.
Sampling distributions: the value of a statistic will in general vary from sample to sample, in which case it will have its own probability distribution, called **sampling distribution**. A statistic is used to estimate the value of a *parameter* θ in a distribution is called an **estimator** (the random variable) or an **estimate** (the value). If $\hat{\theta}$ is an estimator of θ, the mean of its sampling distribution, $E[\hat{\theta}]$, is called the *sampling mean.* The variance, $Var(\hat{\theta})$, is called the *sampling variance.*
$\sqrt{Var(\hat{\theta})}$ is called the *standard error* of $\hat{\theta}$. If $E[\hat{\theta}]=\theta$ then $\hat{\theta}$ is an unbiased estimator of θ, e.g. $\bar{X}$ is an unbiased estimator of μ and has sampling variance σ^2/n where $Var(X_i)=\sigma^2$, $(i=1,2,...,n)$.

Corrected sum of squares

$$S_{xx}=\sum(x_i-\bar{x})^2=\sum x_i^2-n\bar{x}^2=\sum x_i^2-\frac{(\sum x_i)^2}{n}$$

has expectation $(n-1)\sigma^2$ so that dividing S_{xx} by $(n-1)$ will give an unbiased estimator of σ^2, denoted s^2.

Normal and Chi-square distributions

If $X_1,X_2,...,X_n$ are independently and identically $\sim N(\mu,\sigma^2)$ then $\sum(\frac{X_i-\mu}{\sigma})^2 \sim x_n^2$, a Chi-square distribution with n **degrees of freedom**.
Also, $\bar{X} \sim N(\mu,\frac{\sigma^2}{n})$ independently of $\frac{S_{xx}}{\sigma^2} \sim x^2_{(n-1)}$.

Simple Linear Regression To fit the straight line $y=\alpha+\beta x$ to the data $(x_i,y_i), i=1,2,...,n$ by the method of **least squares** the estimates of slope, β, and intercept, α, are given by

$$b = \frac{\sum x_i y_i - \frac{1}{n}(\sum x_i \sum y_i)}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2} = \frac{S_{xy}}{S_{xx}}, \quad a = \bar{y} - b\bar{x}$$

If we assume that the x_i are known and the the y_i have normal distribution with means $\alpha + \beta x_i$, and constant variance σ^2, written as $y_i \sim N(\alpha + \beta x_i, \sigma^2)$, then if x_0 is a fixed value

$$b \sim N(\beta, \frac{\sigma^2}{S_{xx}})$$
$$a \sim N(\alpha, \sigma^2(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}})$$
$$a + bx_0 \sim N(\alpha + \beta x_0, \sigma^2(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}})$$

A common alternative is to use $\hat{\alpha}$ for a and $\hat{\beta}$ for b.

Correlation

Given observations (x_i, y_i), $i = 1, 2, ..., n$ on two random variables X and Y the **Pearson (product moment)** correlation between them is given by

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{xy}}} = \frac{\sum x_i y_i - \frac{1}{n}(\sum x_i \sum y_i)}{\sqrt{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}\sqrt{\sum y_i^2 - \frac{1}{n}(\sum y_i)^2}}$$

We use r to estimate the correlation, ρ, between X and Y. For large n, r is approximately, $\sim N(\rho, \frac{1}{n-2})$. The (**Spearman**) Rank Correlation Coefficient is given by

$$r_S = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$$

where d_i is the difference between the *ranks* of (x_i, y_i), $i = 1, 2, ..., n$. If ranks are tied, see Kotz, S. and Johnson, L. (1988) Encyclopedia of Statistical Sciences.

Time Series

A time series $Y_t, (t = 1, 2, ..., n)$ is a set of n observations recorded through time t, (e.g. days, weeks, months). The arithmetic mean of blocks k successive values

$$\frac{Y_1 + Y_2 + ... + Y_k}{k}, \frac{Y_2 + Y_3 + ... + Y_{k+1}}{k}, ...$$

is a **simple model average** of order k, itself a time series which is *smoother* than Y_t and can be used to track , or estimate, the underlying level, μ_t, of Y_t. If $0 < a < 1$ an **exponentially weighted moving average** (EWMA) at time t uses a discounted weighted average of current and past data to estimate μ_t with

$$\hat{\mu_t} = aY_t + a(1-a)Y_{t-1} + a(1-a)^2 Y_{t-2} + ...$$

This is equivalent to the recurrence relation

$$\hat{\mu_t} = aY_t + (1-a)\hat{\mu_{t-1}}$$

Moving averages are often plotted on the same graph as Y_t. If Y_t additionally contains trend, R_t, the rate of change of data per unit time, and $\mu_t = \mu_{t-1} + R_{t-1}$, then the recurrence relation is

$$\hat{\mu}_t = aY_t + (1-a)(\hat{\mu}_{t-1} + \hat{R}_{t-1})$$

If $0 < b < 1$ the *trend smoothing equation* is

$$\hat{R}_t = b(\hat{\mu}_t - \hat{\mu}_{t-1}) + (1-b)\hat{R}_{t-1}$$

known as *Holt's Linear Exponential Smoothing*. If Y_t also contains *seasonality*, S_t, a smoothing constant γ, $(0 < \gamma < 1)$, is used in *seasonal smoothing equation*, $\hat{S}_t = \gamma Y_t/\hat{\mu}_t + (1-\gamma)\hat{S}_{t-k}$, assuming periodicity is k, with *multiplicative* seasonality. For monthly data $k = 12$.

Forecasting from time n (now) to time $n+h$, $(h = 1, 2, ...)$
Level only, $\hat{Y}_{n+h} = \hat{\mu}_n$ the latest EWMA.
Level and constant trend, $\hat{Y}_{n+h} = a + b(n + h)$, the simple linear regression trend line of Y_t again t.
Level and changing trend, $\hat{Y}_{n+h} = \hat{\mu}_n + h\hat{R}_n$.
Level, changing trend and seasonality $\hat{Y}_{n+h} = \hat{\mu} + h\hat{R}_n$, where $\hat{\mu}_n = aY_n/\hat{S}_{n-12} + (1 - a)(\hat{\mu}_{n-1} + \hat{R}_{n-1})$.

Permutations and combinations
The number of ways selecting r objects out of a total of n, where the order of selection is important, is the number of **permutations**: ${}^n\mathrm{P}_r = \frac{n!}{(n-r)!}$. The number of ways in which r objects can be selected from n when the order of selection is not important is the number of **combinations**: ${}^n\mathrm{C}_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$. ${}^n\mathrm{C}_r$ must equal to 1, so $0! = 1$ and ${}^n\mathrm{C}_0 = 1; {}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$. Also,

$$
{}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + ... + {}^n\mathrm{C}_{n-1} + {}^n\mathrm{C}_n = 2^n
$$
$$
{}^{n+1}\mathrm{C}_r = {}^n\mathrm{C}_r + {}^n\mathrm{C}_{r-1}
$$

Random variables
Data arise from observations on variables that are **measured** on different **scales.** A *nominal* scale is used for named categories (e.g. race, gender) and *ordinal* scale for data that can be ranked (e.g. attitudes, position)- no arithmetic are valid with either. *Interval* scale measurements can be added and subtracted only (e.g. temperature), but with *ratio* scale measurements (e.g. age, weight) multiplication and division can be used meaningfully as well. Generally, random variables are either *discrete* or *continuous.* Note: in reality, all data are discrete because the accuracy of measuring is always limited.
A **discrete** random variable X can take one of the values $x_1, x_2, ...,$, the probabilities $p_i = P(X = x_i)$ must satisfy $p_i \geq 0$ and $p_1 + p_2 + ... = 1$. The pairs (x_i, p_i) form the **probability mass function** (pmf) of X.
A **continuous** random variable X takes values x from a continuous set of possible values. It has **probability density function** (pdf) $f(x)$ that satisfies $f(x) \geq 0$ and $\int f(x)dx = 1$ with $P(a \leq x \leq b) = \int_a^b f(x)dx$.

Expected values
The expected value of a function $H(X)$ of a random variable X is defined as

$$
\begin{cases} \sum H(x_i)P(X = x_i), & X \text{ discrete} \\ \int H(x)f(x)dx, & X \text{ continuous} \end{cases}
$$

Expectation is linear in that the expectation of a linear combination of functions is the same linear combination of expectations. For example,

$$
E[X^2 + logX + 1] = E[X^2] + E[logX] + 1
$$

but

$$
E[logX] \neq logE[X] \text{ and } E[1/X] \neq 1/E[X]
$$

Variance
The variance of a random variable is defined as

$$
Var(X) = E[(X - \mu)^2] = E[X^2] - \mu^2
$$

Properties:
$Var(X) \geq 0$ and is equal to 0 only if X is constant
$Var(aX + b) = a^2Var(X)$, where a and b are constants.

Moment generating functions
The moment generating function (mgf) of a random variable is defined as

$$
M_X(t) = E[exp(tX)] \text{ if this exists}
$$

$E[X^k]$ can be evaluated as the:
(i) coefficient of $\frac{t^r}{r!}$ is the power of expansion of $M_X(t)$

(ii) r-th derivative of $M_X(t)$ evaluated at $t = 0$

Measures of location

The **mean** or **expectation** of the random variable X is $E[X]$, the long-run average of realisations of X. The **mode** is where the pmf or pdf achieves a maximum (if it does so). For a random variable, X, the **median** is such that $P(X \geq median) = 1/2$, so that 50% of values of X occur above and 50% below the median.

Percentiles

x_p is the $100 - p$-th percentile of a random variable X if $P(X \leq x_p) = p$. For example, the 5th percentile, $x_{0.05}$ has 5% of the values smaller than or equal to it. The **median** is the 50-th percentile, the l**lower quartile** is the 25th percentile , the **upper quartile** is the 75th percentile.

Measures of dispersion

The **inner-quartile range** is defined to be the difference between the upper and lower quartiles, $UQ - LQ$. The **standard deviation** is defined as the square root of the variance, $\sigma = \sqrt{Var(X)}$, and is in the same units as the random variable X.

Cumulative Distribution Function

This is defined as a function of any real value t by

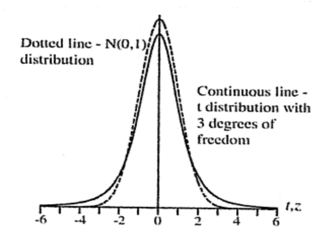
$$F(t) = P(X \leq t)$$

If X is a continuous random variable, F is a continuous function of t; if X is discrete, then F is a step function.

The Central Limit Theorem

If a random sample of size n is taken from *any* distribution with mean μ and variance σ^2, the sampling distribution of the mean will *approximately* $\sim N(\mu, \sigma^2/n)$ where $\sim$ means "is distributed as". The larger the n is, the better the approximation.

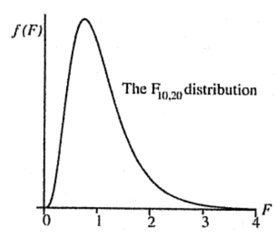
The standard normal and Student's t distribution If a random variable $X \sim N(\mu, \sigma^2)$, $z = (X - \mu)\sigma \sim$



$N(0, 1)$, the *standard normal distribution.* The t distribution with $(n-1)$ degrees of freedom is used in place of z for small samples size n from normal populations when σ^2 is unknown. As n increases the distribution of t converges to $N(0, 1)$. These distributions are used, e.g., for inference about means, differences between means and in regression.

Fisher's F distribution

If $X_1 \sim x^2_{\nu_1}$ and $X_2 \sim x^2_{\nu_2}$ are independent random variables then



$$\frac{X_1/\nu_1}{X_2/\nu_2} \sim F_{\nu_1,\nu_2}$$

the F distribution with (ν_1, ν_2) degrees of freedom. This distribution is used, for example, for inference about the ratio of two variances, in Analysis of Variance (ANOVA), and in simple and multiple linear regression.