# Brownian Motion & Geometric Brownian Motion

Financial Mathematics Clinic

SLAS – University of Kent

These slides are (mainly) aimed to

- Last-stage undergraduate students.
- Postgraduate students.

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Objective

- A brief revision of the definitions and some properties of the Brownian motion and the Geometric Brownian motion.

# Glossary

- *Almost surely (a.s.)*. In Probability, an event is said to happen almost surely, if it happens with probability 1.
- *Stochastic process*. A stochastic process is a collection of random variables indexed in some set, e.g., $\mathbb{N}, \mathbb{R}^+$.
- *Stochastic differential equation (SDE).* An SDE is a differential equation with at least one stochastic component.

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- Louis Bachelier used the BM for the stochastic analysis of the Paris stock exchange (1900).
- Albert Einstein produced a quantitative theory of the BM (1905).
- The BM has an important role in Finances for the modelling of the dynamics of stocks.

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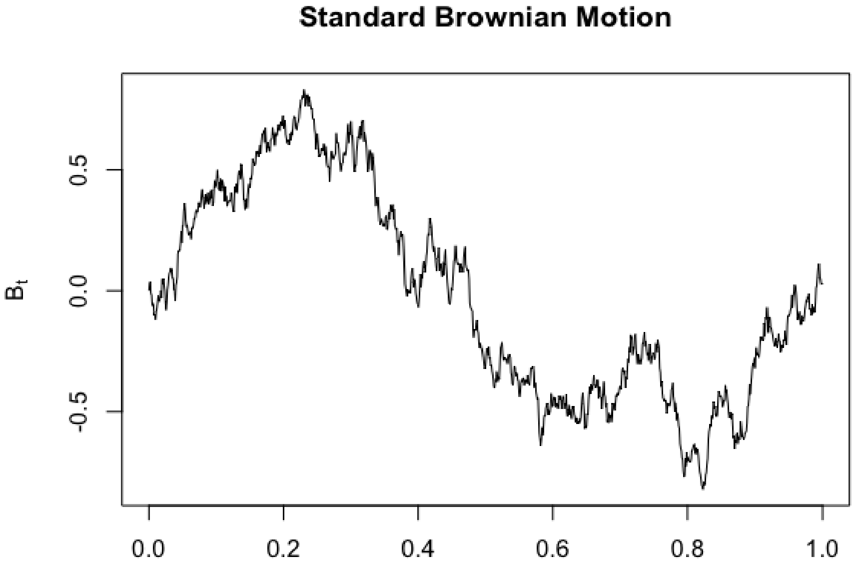
- $B_t \sim N(0, t)\ \forall t > 0$ (**NB. It can be negative!**).

# Graphical example



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Given $\mu$, $\sigma > 0$ and $x_0 > 0$. The GBM is the continuous time process, $X_t$, that solves the SDE

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The solution is given by

$$X_t = x_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right]$$

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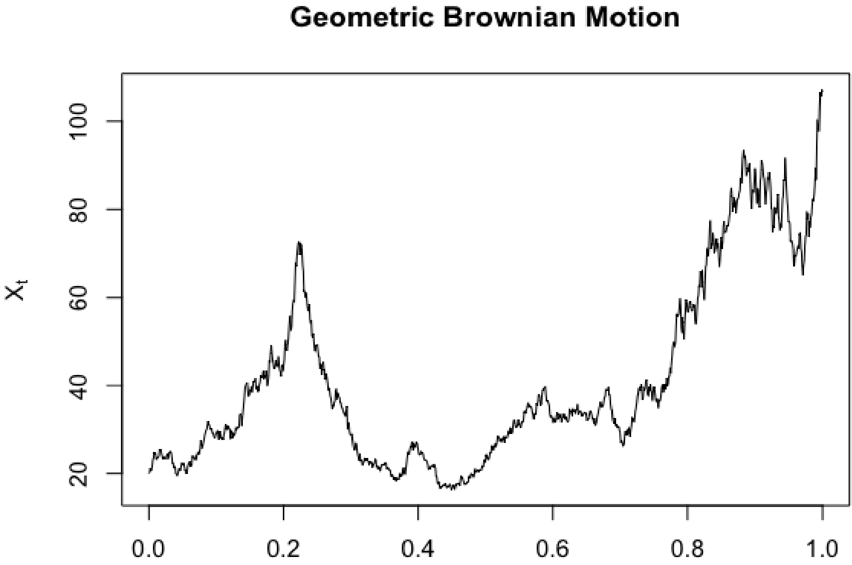

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- Expected returns are independent of the price.
- "Easy" calculations.

Cons

- The constant variance is not realistic.
- Cannot model jumps.

# To book a maths/stats appointment...

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QUESTIONS?