Black-Scholes-Merton

Financial Mathematics Clinic

SLAS – University of Kent

These slides are (mainly) aimed to

- Last-stage undergraduate students.
- Postgraduate students.

Introduction

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Objective

- Discuss the Black-Scholes-Merton model, the assumptions, its limitations and its application to pricing European options.

Glossary

- *No arbitrage*. Absence of opportunities to earn a risk-free profit with no investment.
- *Strike*. A fixed price at which the owner can buy or sell the underlying security.
- *Maturity*. The expiration date of a financial instrument.
- *European options*. Financial derivative that can only be exercised at maturity.
- *Brownian motion and geometric Brownian motion*. Time-continuous stochastic processes used in Finances to model the dynamics of stocks.

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- Pricing of European options.

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- No arbitrage
- Able to borrow and lend any amount using the riskless rate.

The dynamics of the market are described by

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$$

$$B_t = B_0 \exp(rt)$$

where W_t is a standard Brownian motion.

The rational price C_T of a standard European call option with payoff function $f_T = (S_T - K)^+$ is

$$C_T = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where,

- $d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Black-Scholes Formula

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- K is the strike.
- T is the maturity.
- Φ is the normal distribution.

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- Does not capture extreme movements.
- In reality the volatility is **not constant**.
- Suitable only for European options.
- Originally it does not consider dividends.
- Risk-free rates might not be constant.

Black-Scholes Formula with dividends

The rational price C_T of a European call option with payoff function $f_T = (S_T - K)^+$ and paying (constant) divideinds is

$$C_T = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

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- K is the strike.
- T is the maturity.
- Φ is the normal distribution.
- q is the dividend rate.

Example

Stock XYZ is trading for £50 with estimated volatility of 10% and with a risk-free rate of 5%. Calculate the value of a 1-year call and put standard European options with strike £50.

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- Calculate d_1 and d_2.

$$d_1 = \frac{\log\left(\frac{50}{50}\right) + \left(.05 + \frac{.1^2}{2}\right)1}{.1\sqrt{1}} = .55, \quad d_2 = .55 - .1\sqrt{1} = .45$$

Example (continuation)

- Evaluate the normal distribution on d_1 and d_2.

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- Obtain the price of the call using B-S

$$C_1 = 50 * .7088403 - 50 * e^{-.05} * .6736448 = 3.402479$$

- Obtain the price of the put using the *call-put parity*.

$$C_T - P_T = S_T - Ke^{-rT}$$

$$\Rightarrow \quad P_1 = 3.402479 - 50 + 50e^{-.05} = 0.9639502$$

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QUESTIONS?