L'Hopital's Rule - The value \( \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \)

providing that \( f(c) = g(c) = 0 \) or \( f(c) = g(c) = \infty \)

and \( \lim_{x \to c} \frac{f'(x)}{g'(x)} \) exists

Way to Remember - "Differentiate top and bottom until you get something useful!"

Easy Examples

1) Find the limit of \( \frac{\sin(x)}{x} \) as \( x \to 0 \)

Answer: \( \lim_{x \to 0} \frac{\sin(x)}{x} \) does not exist as \( \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} x = 0 \)

but by L'Hopital's rule

\( \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1 \)

2) Find the limit of \( \frac{\cos(x) - \cos(2x)}{1 - \cos(x)} \) as \( x \to 0 \)

Answer: \( \lim_{x \to 0} \frac{\cos(x) - \cos(2x)}{1 - \cos(x)} \) does not exist

as \( \lim_{x \to 0} \frac{\cos(x) - \cos(2x)}{1 - \cos(x)} = \lim_{x \to 0} \frac{1 - \cos(x)}{0} = 0 \)

but by L'Hopital's rule

\( \lim_{x \to 0} \frac{\cos(x) - \cos(2x)}{1 - \cos(x)} = \lim_{x \to 0} \frac{-\sin(x) + 2\sin(2x)}{-\sin(x)} = \lim_{x \to 0} \frac{-\cos(2x) + 4\cos(2x)}{\cos(2x)} \)

\( = \frac{-1 + 4}{1} = 3 \)
1. Evaluate the following limit using L'Hospital's rule:

\[ \lim_{x \to 0} \frac{e^{3x} - 1 - 3x}{e^{2x} - \cos x} \]

a) This limit does not exist

\[ \lim_{x \to 0} \frac{e^{3x} - 1 - 3x}{e^{2x} - \cos x} = e^0 - 1 = 0 \]

and

\[ \lim_{x \to 0} \frac{e^{2x} - \cos x}{e^{2x} - \cos x} = e^0 - 1 = 0 \]

b) but by L'Hospital's rule

\[ \lim_{x \to 0} \frac{\frac{d}{dx} \left( e^{3x} - 1 - 3x \right)}{\frac{d}{dx} \left( e^{2x} - \cos x \right)} = \lim_{x \to 0} \frac{3e^{3x} - 3}{2e^{2x} + \sin x} = \frac{\lim_{x \to 0} \frac{3e^{3x}}{2e^{2x} + \sin x}}{\lim_{x \to 0} \frac{2e^{2x} + 4xe^{2x} - \cos x}{2e^{2x} + 4xe^{2x} - \cos x}} = \frac{3}{2 + 0 + 1} = \frac{3}{3} = 1 \]

2. Evaluate the following limit using L'Hospital's rule:

\[ \lim_{x \to 0} \frac{\sin(x^2)}{x - \log(\cos x)} \]

a) This limit does not exist as

\[ \lim_{x \to 0} \sin(x^2) = \sin(0) = 0 \]

and

\[ \lim_{x \to 0} \log(\cos x) = \log(\cos(0)) = \log(1) = 0 \]

b) but by L'Hospital's rule

\[ \lim_{x \to 0} \frac{\frac{d}{dx} \sin(x^2)}{\frac{d}{dx} \left( x - \log(\cos x) \right)} = \lim_{x \to 0} \frac{2x \cos(x^2)}{\tan(x)} \]

hence

\[ \lim_{x \to 0} \frac{\sin(x^2)}{x - \log(\cos x)} = \lim_{x \to 0} \frac{2x \cos(x^2)}{\tan(x)} = \lim_{x \to 0} \frac{2 \cos(x^2) - 4x^2 \sin(x^2)}{1 + \tan^2(x)} = \frac{2}{1} = 2 \]
Evaluate the following limit using L'Hospital's rule:

\[
\lim_{x \to 0} \frac{\log(1 + x) - x}{\cos x - 1}.
\]

\[\frac{1}{\cos(x)} \cdot \frac{\frac{1}{1+x} - 1}{-\sin(x)} = \frac{-1}{-1} = 1\]
Repeated integrals work in much the same way as partial derivatives.

Way to remember: work outwards.

\[ \int_1^e \int_0^{\sqrt{x}} \frac{2y \log x \, dy \, dx}{x} \]

Q/ Evaluate the following repeated integral:

A/ \[ e \int_0^{\sqrt{x}} 2y \left( \frac{\log x}{2x} \right) \, dy \, dx = e \int_0^{\sqrt{x}} (\log x) \left( \int_0^y dy \right) \, dx \]

\[ = e \int_0^{\sqrt{x}} \frac{\log x}{2x} \left[ y^2 \right]_0^x \, dx \]

\[ = e \int_0^{\sqrt{x}} \frac{\log x \cdot (x - 0)}{2x} \, dx \]

\[ = e \int_0^{\sqrt{x}} \log x \, dx = \left[ \frac{\log x}{x} \right]_0^e \]

\[ = \frac{1}{e} - 1 \]
1/ Evaluate the following repeated integral:

\[ \int_0^\pi/2 \int_y^{\pi/2} \cos y \sin x \, dx \, dy \]

\[ = \pi \int_0^{\pi/2} \cos(y) \left( \int_y^{\pi/2} \sin(x) \, dx \right) \, dy \]

\[ = \pi \int_0^{\pi/2} \cos(y) \left[ \frac{\cos(x)}{2} \right]_y^{\pi/2} \, dy \]

\[ = \frac{\pi}{2} \int_0^{\pi/2} \cos^2(y) \, dy \]

\[ = \frac{\pi}{2} \left[ \frac{1}{2} y + \frac{\sin(2y)}{2} \right]_0^{\pi/2} = \frac{\pi}{4} \]

2/ Evaluate the following repeated integral:

\[ \int_0^{\pi/2} \int_0^x \cos x \cos y \, dy \, dx \]

**Answer** = \( \frac{1}{2} \)
a) Integrate \( \int_{-\infty}^{\infty} x^2 e^{xt} \, dt \, dx \)

\[
= \int_{-\infty}^{\infty} x^2 \left( \int_{-\infty}^{\infty} e^{xt} \, dt \right) \, dx
\]

must treat \( x \) as a constant when integrating. Same as \( \int e^{kt} \, dt = \frac{e^{kt}}{k} + c \)

\[
= \int_{-\infty}^{\infty} x^2 \left[ \frac{e^{xt}}{x} \right]_{-\infty}^{\infty} \, dx
\]

\[
= \int_{-\infty}^{\infty} x^2 \, dx
\]

\[
= \left[ \frac{e^{x^2}}{2} \right]_{-\infty}^{\infty}
\]

\[
= \frac{e^2}{2} - \frac{e^{-2}}{2} = 0
\]

A **TRICK WORTH REMEMBER**

\( \int x e^{x^2} \, dx = \frac{e^{x^2}}{2} + c \)

**WHY?** As \( \frac{d}{dx}(\frac{e^{x^2}}{2} + c) = \frac{1}{2} \cdot 2x \cdot e^{x^2} = xe^{x^2} \)

\( \int x^2 e^{x^3} \, dx = \frac{e^{x^3}}{3} + c \)

\( \int x \cos(x^2) \, dx = \frac{\sin(x^2)}{2} + c \)
Calculus - Local Minima and Maxima
17 March 2010
13:21

Rules for Local Maxima/Minima and Saddle Points

• Take a function \( f(x, y) \) at a point \((a, b)\)

• First Condition \( \frac{\partial^2 f}{\partial x^2}(a,b) = \frac{\partial^2 f}{\partial y^2}(a,b) = 0 \)

• Second Condition

Let \( D = \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \)

if

• \( D|_{(a,b)} > 0 \) and \( \frac{\partial^2 f}{\partial x^2} > 0 \) \hspace{1cm} \text{Minimum}

• \( D|_{(a,b)} > 0 \) and \( \frac{\partial^2 f}{\partial x^2} < 0 \) \hspace{1cm} \text{Maximum}

• \( D|_{(a,b)} < 0 \) \hspace{1cm} \text{Saddle}

• \( D = 0 \)
Calculus - Local Minima and Maxima Example

If \( f(x, y) = 2 \cos(x + y) + \sin(xy) \), determine \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y} \) and \( \frac{\partial^2 f}{\partial y^2} \). Hence show that \( f \) has a local maximum at the origin \((0, 0)\). [15 marks]

\[
\frac{\partial f}{\partial x} = -2 \sin(x+y) + 4 \cos(xy) \\
\frac{\partial f}{\partial y} = -2 \sin(x+y) + x \cos(xy)
\]

\[
\frac{\partial^2 f}{\partial x^2} = -2 \cos(x+y) - 4 \sin(xy) \\
\frac{\partial^2 f}{\partial y^2} = -2 \cos(x+y) - x^2 \sin(xy)
\]

\[
\frac{\partial^2 f}{\partial x \partial y} = -2 \cos(x+y) - xy \sin(xy) + \cos(xy)
\]

\[
\frac{\partial^2 f}{\partial x \partial y} \bigg|_{(0,0)} = -1
\]

**Conditions for Local Maxima**

1. \( \frac{\partial f}{\partial x} \bigg|_{(0,0)} = \frac{\partial f}{\partial y} \bigg|_{(0,0)} = 0 \) \( \text{ SHOWN ABOVE} \)

2. \( \frac{\partial^2 f}{\partial x^2} \bigg|_{(0,0)} < 0 \) \( \text{ SHOWN ABOVE} \)

3. \( \left( \frac{\partial^2 f}{\partial x \partial y} \bigg|_{(0,0)} \right)^2 - \left( \frac{\partial^2 f}{\partial x^2} \bigg|_{(0,0)} \right) \left( \frac{\partial^2 f}{\partial y^2} \bigg|_{(0,0)} \right) > 0 \)

\[
(-2) \times (-2) - 1 = 3 > 0 \quad \text{CORRECT}
\]

Hence \( \text{IS A LOCAL MAXIMA} \)
If \( f(x, y) = \cosh(x + 2y) + \sin(x^2 - y^2) \), determine \( \frac{\partial f}{\partial x} \), \( \frac{\partial f}{\partial y} \), \( \frac{\partial^2 f}{\partial x^2} \), \( \frac{\partial^2 f}{\partial x \partial y} \) and \( \frac{\partial^2 f}{\partial y^2} \). Hence show that \( f \) has a local minimum at the origin \((0,0)\). 

\[
\frac{\partial f}{\partial x} = \sinh(x+2y) + 2x\cos(x^2-y^2) \quad \frac{\partial^2 f}{\partial x^2} = \cosh(x+2y) - 4x^2 \sin(x^2-y^2) + 2 \cos(x^2-y^2)
\]

\[
\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0
\]

\[
\left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} = 3
\]

\[
\frac{\partial f}{\partial y} = 2 \sinh(x+2y) - 2y \cos(x^2-y^2) \quad \frac{\partial^2 f}{\partial y^2} = 4 \cosh(x+2y) - 4y^2 \sin(x^2-y^2) - 2 \cos(x^2-y^2)
\]

\[
\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = 0
\]

\[
\left. \frac{\partial^2 f}{\partial y^2} \right|_{(0,0)} = 2
\]

\[
\frac{\partial^2 f}{\partial x \partial y} = 2 \cos(x+2y) + 4xy \sin(x^2-y^2)
\]

\[
\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} = 2
\]

Conditions for Local Minimum:

1. \( \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0 \) shown above \( \checkmark \)

2. \( \left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} > 0 \) shown above \( \checkmark \)

3. \( \left( \left. \frac{\partial^2 f}{\partial x^2} \right|_{(0,0)} \right) \left( \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(0,0)} \right) - \left( \left. \frac{\partial^2 f}{\partial y^2} \right|_{(0,0)} \right)^2 > 0 \)

\[
3 \times 2 - 4 = 2 > 0 \quad \text{true} \quad \checkmark
\]

so have a local minima
You will be asked to show that

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \]

then

\[ \frac{\partial^4 f}{\partial x^4} - \frac{\partial^4 f}{\partial y^4} = 0 \]

You should write down exactly the following

1. \[ \frac{\partial^2}{\partial x^2} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) = \frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial x^2 \partial y^2} = 0 \]

2. \[ \frac{\partial^2}{\partial y^2} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) = \frac{\partial^4 f}{\partial y^4} + \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0 \]

Therefore

\[ \frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0 \]

\[ \Rightarrow \frac{\partial^4 f}{\partial x^4} - \frac{\partial^4 f}{\partial y^4} = 0 \quad \Box \]
TREAT AS WALKING ALONG A LANDSCAPE
WALK LEFT → RIGHT
WALKING UPHILL
= MORE EFFORT
= POSITIVE DERIVATIVE
WALKING DOWNHILL
= LESS EFFORT
= NEGATIVE DERIVATIVE
WALKING FLAT
= ZERO DERIVATIVE

HERE WE ARE ALWAYS WALKING DOWNHILL NEAR THE TURNING POINT

THE DERIVATIVE OF \( f'(x) \)
(Which is \( f''(x) \)) IS THEREFORE NEGATIVE
Calculus - Hyperbolic Trig Functions

21 March 2010
15:52

Definition
\[
\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}
\]

Derivatives
\[
\frac{d}{dx} \sinh(x) = \cosh(x)
\]
\[
\frac{d}{dx} \cosh(x) = \sinh(x)
\]

Asymptotic Behaviour
\[
\lim_{x \to \pm \infty} \sinh(x) = \pm \infty
\]
\[
\lim_{x \to \pm \infty} \cosh(x) = \infty
\]

O Behaviour
\[
\sinh(0) = 0
\]
\[
\cosh(0) = 1
\]
This technique allows you to find the derivatives of weird looking equations.

Normally you will be asked to find the explicit derivative of a function i.e.
\[ y = x^3 + \sin(2xe^x) \]

Implicit differentiation deals with equations of the form
\[ F(x, y) = F(x, y) \] which may not be written
\[ y = f(x) \]

There is a simple rule for remembering how to do these.

"Differentiate x normally and .... differentiate y normally but multiply by dy/dx"

or
\[ f(x) \rightarrow f'(x) \]
\[ g(y) \rightarrow g'(y) \cdot \frac{dy}{dx} \]

**WHY THIS WORKS**

Consider (from Wikipedia) the derivative of the function
\[ x^4 + 2y^2 = 8 \]

**Implicit**
\[ 4x^3 + 4y \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{-x^3}{y} \]

**Direct**
\[ y = \pm \left( \frac{8 - x^4}{2} \right)^{1/2} \]
\[ \frac{dy}{dx} = \frac{\pm(1)(-2x^3)(8-x^4)^{-1/2}}{2} \]
\[ = \frac{\pm(-x^3)}{(8-x^4)^{1/2}} \]

*But from* \[ y = \pm \left( \frac{8 - x^4}{2} \right)^{1/2} \]
\[ = \frac{-x^3}{y} \]

**EASY**

**HORRIBLE**
Complete the following. Where possible also use the direct (HORRIBLE) method from the previous page until you are convinced that implicit differentiation is much easier.

41. \( y^3 + x \cos(x) = x^2 \)

21. \( xy^2 = e^x \) (must use chain rule on LHS)

31. \( e^{y \cos(x)} = x^3 \sin(x) \)

41. \( e^{y^2 \cos(x)} = x^3 \sin(x) \)

The function \( y \) of \( x \) satisfies the following equation: \( \log y = -\frac{y}{x} + xy^2 \).

Calculate \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).
1/ The function \( y \) of \( x \) satisfies the following equation: \( e^y = x^2 - \frac{x}{y} + xy^3 \).

Calculate \( \frac{dy}{dx} \) in terms of \( x \) and \( y \). Find a pair of real numbers \( x, y \) which satisfy the equation above.

\[
\frac{dy}{dx} e^y = 2x - \frac{1}{y} + x \frac{dy}{dx} + y^3 + 3xy^2 \cdot \frac{dy}{dx}
\]

We then take all \( \frac{dy}{dx} \) terms to one side.

\[
\frac{dy}{dx} (e^y - \frac{x}{y} - 3xy^2) = 2x - \frac{1}{y} + y^3
\]

\[
\frac{dy}{dx} = \frac{2x - \frac{1}{y} + y^3}{e^y - \frac{x}{y} - 3xy^2}
\]

Second part: let \( y = 1 \) \( \Rightarrow \) \( x = e^{1/2} \)

2/ The function \( y \) of \( x \) satisfies the following equation: \( y^2 = \cos(xy) + \log(x + y) \).

Calculate \( \frac{dy}{dx} \) in terms of \( x \) and \( y \). Find a pair of real numbers \( x, y \) which satisfy the equation above.

\[
2y \cdot \frac{dy}{dx} = -\sin(xy) \cdot \text{diff}(xy) + \frac{1}{x+y} \cdot \text{diff}(x+y)
\]

\[
2y \cdot \frac{dy}{dx} = -\sin(xy) \cdot (y + x \frac{dy}{dx}) + \frac{1}{(x+y)} \cdot (1 + \frac{dy}{dx})
\]

\[
\frac{dy}{dx} \left(2y + \sin(xy) \cdot x - (x+y)^{-1}\right) = -y \sin(xy) + (x+y)^{-1}
\]

\[
\frac{dy}{dx} = \frac{(x+y)^{-1} - y \sin(xy)}{2y + \sin(xy) \cdot x - (x+y)^{-1}}
\]
Method
All of these may be proven with 3 steps

1 | SETUP AND REVERSE
2 | IMPLICIT DERIVATIVE
3 | SUBSTITUTION

Claim \(- \frac{d}{dx} (\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}\)

Step 1 \( y = \arcsin(x) \quad \Rightarrow \quad x = \sin(y) \)

Step 2 \( \frac{d}{dx} (x) = \frac{d}{dc} (\sin(y)) \)

- Recall from Step 1 that \( y = y(x) \) and so must use implicit differentiation

\( 1 = \cos(y) \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)} \)

Step 3 • Recall that \( \sin^2(y) + \cos^2(y) = 1 \)

- and so \( \cos(y) = \sqrt{1 - \sin^2(y)} \)

- But from Step 1 \( \sin(y) = x \) and so \( \cos(y) = \sqrt{1 - x^2} \)

- And \( \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \)

Exercise
Prove that \( \frac{d}{dx} (\arctan(x)) = \frac{1}{1 + x^2} \)

Hint - you will need to know that
\( \frac{d}{dy} (\tan(y)) = \sec^2(y) \) and \( \sec^2(y) = 1 + \tan^2(y) \)

Use quotient rule on\( f(y) = \frac{\sin(y)}{\cos(y)} \)

Use \( \sin^2(y) + \cos^2(y) = 1 \)

and \( \sec^2(y) = \frac{1}{\cos^2(y)} \)
Want to Remember - "Top and bottom must cancel"

Suppose we want to differentiate

\[ f(x,y) = f(x(t), y(t)) \]

with respect to \( t \)

Then

\[ \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} \]

\( \leftarrow \) "top and bottom cancel"

\[ \frac{df}{dx} = e^x \sin(y) + y^2 e^x \]

where \( x = \sin(t) \) and \( y = t^3 \)

\[ \frac{df}{dy} = e^x \cos(y) + 2y e^x \]

At

\[ \frac{df}{dx} = e^{\sin(t)} \sin(t^3) + t^2 e^{\sin(t)} \]

\[ \frac{df}{dy} = e^{\sin(t)} \cos(t^3) + 2t^3 e^{\sin(t)} \]

\[ x = \sin(t) \]

\[ y = t^3 \]

\[ \frac{dx}{dt} = \cos(t) \]

\[ \frac{dy}{dt} = 3t^2 \]

\[ \frac{df}{dt} = \cos(t) \left( e^{\sin(t)} \sin(t^3) + t^2 e^{\sin(t)} \right) \]

\[ + 3t^2 \left( e^{\sin(t)} \cos(t^3) + 2t^3 e^{\sin(t)} \right) \]
If \( f(x, y) = \log(x - y) + \cos(xy) + y^2e^x \), determine \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \). Find \( \frac{df}{dt} \) in terms of \( x \) and \( y \) where \( x = \cos t, \ y = \sin t \) using the chain rule formula. [10 marks]

\[
\frac{df}{dt} = \frac{1}{x-y} - y\sin(xy) + y^2e^x
\]

\[
\frac{df}{dt} = \frac{1}{x-y} - x\sin(xy) + 2y^2e^x
\]

\[
\frac{df}{dt} = \frac{1}{\cosh t - \sinh t} - \sinh t \cdot \sin (\sinh t \cdot \cos t) + \sin^2 t e^{\cos t}
\]

\[
\frac{df}{dt} = \frac{-1}{\cosh t - \sinh t} - \cos t \cdot \sin (\sinh t \cdot \cos t) + \cos^2 t e^{\cos t}
\]

\[
\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t
\]

\[
\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}
\]

[Q2] If \( f(x, y) = \log(x + y) + \tan(xy) + e^{x^2 - y^2} \), determine \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \). Find \( \frac{df}{dt} \) in terms of \( x \) and \( y \) where \( x = \cosh t, \ y = \sinh t \) using the chain rule formula. [10 marks]

[Q3] If \( f(x, y) = \log(x + y) + \sin(xy) + y^2e^x \), determine \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \). Find \( \frac{df}{dt} \) in terms of \( x \) and \( y \) where \( x = \cos t, \ y = \sin t \) using the chain rule formula. [10 marks]
Recall that \((1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

where \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\)  \((0! = 1)\)

Consequently \((1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\)

- This only works for integer \(n\).
- What about non-integer \(n\)?

\[(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + ...\]

Proof that \(\frac{d}{dx} (x^n) = nx^{n-1}\)

\[(x+h)^n = x^n \left(1 + \frac{h}{x}\right)^n\]

\[= x^n \left(1 + \frac{nh}{x} + \frac{n(n-1)h^2}{2!x^2} + \frac{n(n-1)(n-2)h^3}{3!x^3} + ...\right)\]

\[= x^n + nhx^{n-1} + \frac{n(n-1)h^2}{2!}x^{n-2} + \frac{n(n-1)(n-2)h^3}{3!}x^{n-3} + ...\]

so \((x+h)^n - x^n = \frac{nhx^{n-1} + \frac{n(n-1)h^2}{2!}x^{n-2} + \frac{n(n-1)(n-2)h^3}{3!}x^{n-3} + ...}{2!}

\[\left(\frac{x+h}{h}\right)^n - x^n = \frac{nx^{n-1} + \frac{n(n-1)h}{2!}x^{n-2} + \frac{n(n-1)(n-2)h^2}{3!}x^{n-3} + ...}{h}\]

\[\lim_{h \to 0} \left(\frac{x+h}{h}\right)^n - x^n = nx^{n-1} + o \rightarrow \ldots\]

\[= nx^{n-1}\]
We can use this to find derivatives explicitly.

**Example**

Use the definition of the derivative, 
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \]
to show that:

(i) if \( f(x) = x^{-2} \) then \( f'(x) = -2x^{-3} \).

\[ f'(x) = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{1}{h(x + h)^2} - \frac{1}{x^2} = \lim_{h \to 0} \frac{x^2 - (x + h)^2}{h(x + h)^2} \]
\[ = \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x + h)^2} = \lim_{h \to 0} \frac{-2xh - h^2}{hx^2(x + h)^2} \]
\[ = \lim_{h \to 0} \frac{-2x - h}{x^2(x + h)^2} = \frac{-2x}{x^3} = \frac{-2}{x^2} \]

**Example**

Show that if \( f(x) = x^{3/2} \) then \( f'(x) = \frac{3}{2} x^{1/2} \).

This is found by writing out proof on previous page and then letting \( n = \frac{3}{2} \).
Calculus - Differentiation from First Principles Examples

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(a) Use the definition of the derivative, \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \), to show that:

(i) if \( f(x) = \sqrt{x} \) then \( f'(x) = \frac{1}{2\sqrt{x}} \); [10 marks]

(ii) if \( f(x) = \sin x \) then \( f'(x) = \cos x \). [10 marks]

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(i) Use general proof for \( \frac{d}{dx} (x^n) = nx^{n-1} \) and let \( n = \frac{1}{2} \)

(i) \( f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \)

\[ = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \]

\[ = \sin(x) \left( \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right) + \cos(x) \left( \lim_{h \to 0} \frac{\sin(h)}{h} \right) \]

By L'Hopital's Rule (on both limits)

\[ = \sin(x) \left( \lim_{h \to 0} \frac{-\sin(h)}{h} \right) + \cos(x) \left( \lim_{h \to 0} \frac{\cos(h)}{1} \right) \]

\[ = \sin(x) \times 0 + \cos(x) \times 1 = \cos(x) \]
(b) An isosceles triangle is to be constructed with perimeter (the sum of the lengths of its sides) equal to one metre in such a way as to maximize the area of the triangle. Determine its area. [10 marks]

\[
\begin{align*}
2a + b &= 1 \\
\rightarrow b &= 1 - 2a
\end{align*}
\]

Area = \(ab\) = \(a - 2a^2\)

\[
\frac{f(a)}{2} = a - 2a^2
\]

\[
\frac{f'(a)}{2} = 1 - 4a
\]

For max/min \(\Rightarrow f'(a) = 0 \Rightarrow a = \frac{1}{4}\)

For max \(\Rightarrow f''(a) < 0 \checkmark\)

\[
f(a) = \frac{1}{4} - 2\left(\frac{1}{16}\right) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \text{ m}^2
\]
A right-angled triangle is to be constructed with hypotenuse (the longest side) of length one metre in such a way as to maximize the area of the triangle. Determine the lengths of the other two sides (Hint: Consider the area as a function of the length of one of the sides).

\[ \text{Area} = \frac{ab}{2} \]

By Pythagoras, \(a^2 + b^2 = 1\) \implies \(a = \pm (1-b^2)^{\frac{1}{2}}\)

\[ f(b) = \text{Area} = \frac{b}{2} (1-b^2)^{\frac{1}{2}} \]

For max/min, \(f'(b) = 0 \implies \left( \frac{2b^2 - 1}{2(1-b^2)^{\frac{3}{2}}} \right) = 0 \implies b = \frac{1}{\sqrt{2}} \]

So \(b = \frac{1}{\sqrt{2}}\) and \(a = \frac{1}{\sqrt{2}}\)

An isosceles triangle is inscribed in a circle with radius one metre in such a way as to maximize the area of the triangle. Determine its area. (Hint: Consider the area as a function of the distance from the centre of the circle to the base of the triangle).