SYMPLECTIC MONGE-AMPÈRE EQUATIONS: GEOMETRY AND INTEGRABILITY

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Plan:

- Symplectic Monge-Ampère equations. Equivalence group. Integrability
- Linearisability of symplectic Monge-Ampère equations in 2D
- Linearisability of integrable symplectic Monge-Ampère equations in 3D
- Symplectic Monge-Ampère equations in 4D. Necessary conditions for integrability. Classification. Geometry

References

E.V. Ferapontov, L. Hadjikos and K.R. Khusnutdinova, Integrable equations of the dispersionless Hirota type and hypersurfaces in the Lagrangian Grassmannian, International Mathematics Research Notices, (2010) 496-535; arXiv: 0705.1774 (2007).

B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, Journal of Geometry and Physics, **60** *(2010) 1604-1616.*

Symplectic Monge-Ampère equations

Let $U = (u_{ij})$ be the Hessian matrix of a function $u(x^1, ..., x^n)$. Symplectic Monge-Ampère equations are linear combinations of all possible minors of U

Examples:

first heavenly equation	$u_{13}u_{24} - u_{14}u_{23} = 1$
second heavenly equation	$u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$
Husain equation	$u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$
6D heavenly equation	$u_{15} + u_{26} + u_{13}u_{24} - u_{14}u_{23} = 0$
Special Lagrangian 3 – folds	$Hess\, u = \triangle u$
Affine spheres	$Hess\, u = 1$

Equivalence group $Sp(2n)$ acts by linear transformations of $x^1, ..., x^n, u_1, ..., u_n$

Integrability? Classification? Geometry?

Special Lagrangian 3-folds

Consider the space $\mathbb{C}^3$ with coordinates z^1, z^2, z^3 $(z^k = x^k + iu_k)$

Symplectic form $\omega = du_1 \wedge dx^1 + du_2 \wedge dx^2 + du_3 \wedge dx^3$

Holomorphic volume form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$

$$Im\,\Omega = -du_1 \wedge du_2 \wedge du_3 + du_1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge du_2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge du_3$$

Special Lagrangian 3-folds are specified by the equations $\omega = Im\,\Omega = 0$

In general: symplectic space $\mathbb{R}^{2n}$ with coordinates $x^1, ..., x^n, u_1, ..., u_n$

Symplectic form $\omega = du_1 \wedge dx^1 + ... + du_n \wedge dx^n$

Constant coefficient differential n-form Φ in dx^k, du_k

Symplectic Monge-Ampère equations are specified by the equations $\omega = \Phi = 0$

Manifestly $Sp(2n)$ invariant

Integrability: the method of hydrodynamic reductions

Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0$$

Consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N)$$

The phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Commutativity conditions: $\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$

Definition

A quasilinear system is said to be integrable if, for any number of phases N, it possesses infinitely many hydrodynamic reductions parametrized by N arbitrary functions of one variable.

Example of dKP

$$u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy}$$

First order (hydrodynamic) form, set $v = u_{xx},\ w = u_{xy}$:

$$v_t - vv_x = w_y, \quad v_y = w_x$$

N-phase solutions: $v = v(R^1, ..., R^N),\ w = w(R^1, ..., R^N)$ where

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Then

$$\partial_i w = \mu^i \partial_i v, \qquad \lambda^i = v + (\mu^i)^2$$

Equations for $v(R)$ and $\mu^i(R)$ (Gibbons-Tsarev system):

$$\partial_j \mu^i = \frac{\partial_j v}{\mu^j - \mu^i}, \qquad \partial_i \partial_j v = 2\frac{\partial_i v \partial_j v}{(\mu^j - \mu^i)^2}$$

In involution! General solution depends on N arbitrary functions of one variable.

Generalized dKP

$$u_{xt} - f(u_{xx}) = u_{yy}$$

First order (hydrodynamic) form, set $v = u_{xx},\ w = u_{xy}$:

$$v_t - f(v)v_x = w_y, \quad v_y = w_x$$

N-phase solutions: $v = v(R^1, ..., R^N),\ w = w(R^1, ..., R^N)$ where

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x$$

Then

$$\partial_i w = \mu^i \partial_i v, \qquad \lambda^i = f'(v) + (\mu^i)^2$$

Generalized Gibbons-Tsarev system:

$$\partial_j \mu^i = f''(v)\frac{\partial_j v}{\mu^j - \mu^i}, \qquad \partial_i\partial_j v = 2f''(v)\frac{\partial_i v \partial_j v}{(\mu^j - \mu^i)^2}$$

Involutivity $\longleftrightarrow$ $f''' = 0$

Transformation to quasilinear form

2D Monge-Ampère equation

$$u_{11}u_{22} - u_{12}^2 = 1$$

Set $v = u_{11},\ w = u_{12}$. Then $u_{22} = (1 + w^2)/v$, and we get

$$v_2 = w_1, \quad w_2 = \left((1 + w^2)/v\right)_1$$

3D Monge-Ampère equation

$$u_{11}u_{23} - u_{12}u_{13} = 1$$

Set $v = u_{11},\ w = u_{12},\ r = u_{13}$. Then $u_{23} = (1 + wr)/v$, and we get

$$v_2 = w_1, \quad v_3 = r_1, \quad w_3 = r_2, \quad w_3 = ((1 + wr)/v)_1$$

Symplectic Monge-Ampère equations in 2D

Hessian matrix

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix}$$

Symplectic Monge-Ampère equations are of the form

$$M_2 + M_1 + M_0 = 0$$

Explicitly,

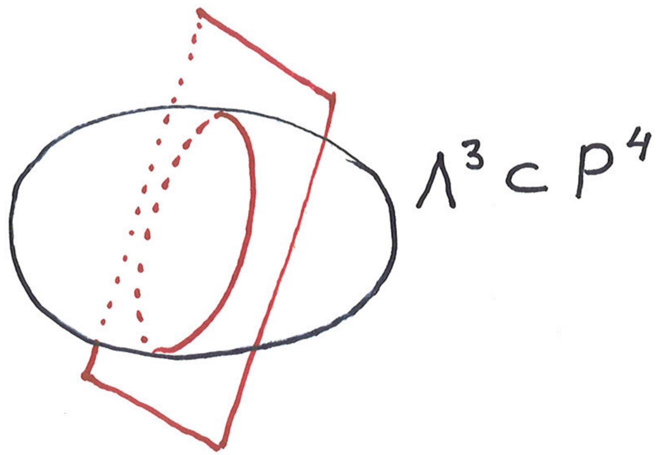
$$\epsilon(u_{11}u_{22} - u_{12}^2) + au_{11} + bu_{12} + cu_{22} + d = 0$$

Any such equation is linearisable by a transformation from $Sp(4)$

Geometry in 2D

Symplectic space with coordinates x^1, x^2, u_1, u_2. Lagrangian planes form the Lagrangian Grassmannian Λ^3

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$



Plücker embedding of Λ^3 in P^4 is $(1 : u_{11} : u_{12} : u_{22} : u_{11}u_{22} - u_{12}^2)$

Symplectic Monge-Ampère equations $\longleftrightarrow$ hyperplanes in P^4

Symplectic Monge-Ampère equations in 3D

Hessian matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}$$

Symplectic Monge-Ampère equations are of the form

$$M_3 + M_2 + M_1 + M_0 = 0$$

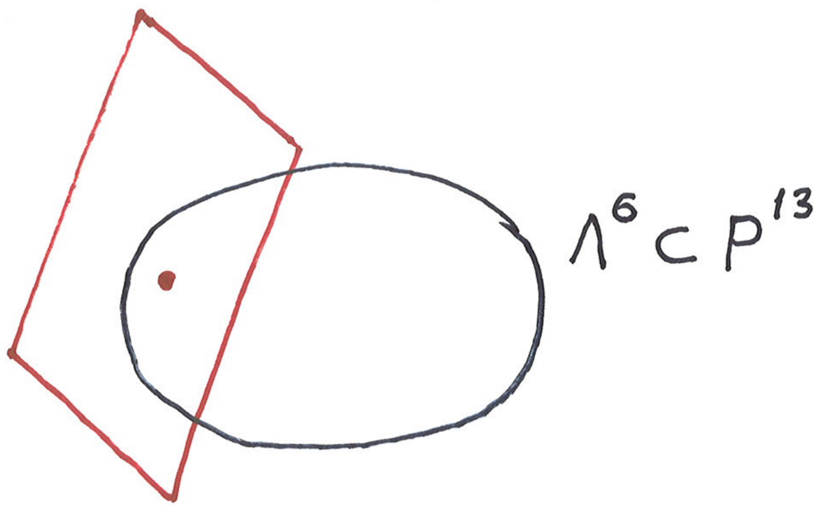
In 3D there exist three essentially different canonical forms modulo the equivalence group $Sp(6)$ (Lychagin, Rubtsov, Chekalov, Banos):

$$u_{11} = u_{22} + u_{33}, \qquad Hess\, u = \triangle u, \qquad Hess\, u = 1$$

Integrability $\longleftrightarrow$ linearisability (not true in dim 4 and higher).

Geometry in 3D

Symplectic space with coordinates $x^1, x^2, x^3, u_1, u_2, u_3$. Lagrangian planes form the Lagrangian Grassmannian Λ^6



Plücker embedding of Λ^6 in P^{13}

Integrable symplectic Monge-Ampère equations $\longleftrightarrow$ hyperplanes tangential to $\Lambda^6 \subset P^{13}$

Symplectic Monge-Ampère equations in 4D

Consider a symplectic Monge-Ampère equation in 4D for $u(x^1, x^2, x^3, x^4)$ and take its traveling wave reduction to 3D,

$$u = u(x^1 + \alpha x^4, x^2 + \beta x^4, x^3 + \gamma x^4) + Q(x, x),$$

where $Q(x, x)$ is an arbitrary quadratic form.

Integrability in 4D $\longleftrightarrow$ linearisability of all traveling wave reductions to 3D

In particular, all traveling wave reductions of the first heavenly equation $u_{13}u_{24} - u_{14}u_{23} = 1$ are linearisable.

Classification of integrable equations in 4D

linear wave	$u_{11} - u_{22} - u_{33} - u_{44} = 0$
first heavenly	$u_{13}u_{24} - u_{14}u_{23} = 1$
second heavenly	$u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$
modified heavenly	$u_{13} = u_{12}u_{44} - u_{14}u_{24}$
Husain	$u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$
general heavenly	$\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0, \quad \alpha + \beta + \gamma = 0$

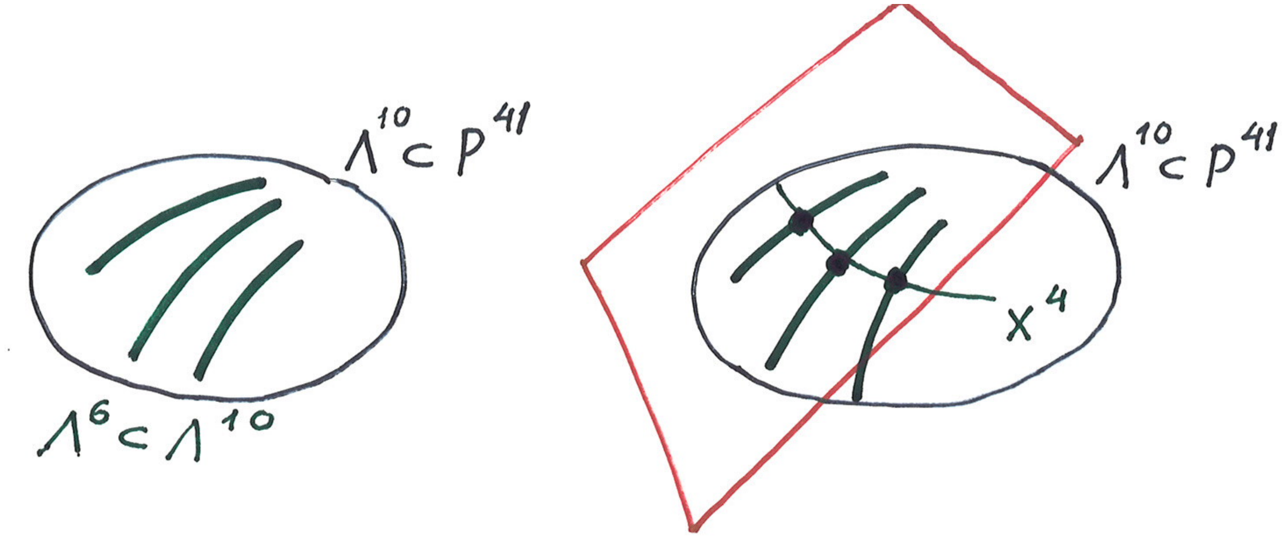
Conjecture

In dimensions $D \geq 4$, any integrable equation of the form $F(u_{ij}) = 0$ is necessarily of the symplectic Monge-Ampère type

Not true in 3D: take the dKP equation $u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy}$

Geometry in 4D

Symplectic space with coordinates $x^1, x^2, x^3, x^4, u_1, u_2, u_3, u_4$. Lagrangian planes form the Lagrangian Grassmannian Λ^{10}



Plücker embedding of Λ^{10} in P^{41} is covered by a 7-parameter family of Λ^6

Integrable symplectic Monge-Ampère equations $\longleftrightarrow$ hyperplanes tangential to $\Lambda^{10} \subset P^{41}$ along a four-dimensional subvariety X^4 which meets all $\Lambda^6 \subset P^{41}$