A PEDAGOGICAL NOTE ON THE LONG RUN OF MACRO ECONOMIC MODELS

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Abstract

This paper surveys and motivates the reasons for incorporating explicit long run or supply features into applied estimated macro economic models. It defines some basic stability conditions for such models and illustrates elements of long run considerations with reference to a small (22 equation) prototype model which fashions itself after many current models. It also shows that many of the key decisions when setting up the long run extended simulation horizon reduce to a relatively small number of relationships.

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1. Introduction

Many macroeconomic models possess an operational long run or steady state version. In this (mainly pedagogical) paper we discuss why modellers incorporate explicit long run properties into their models and the consequences and trade-offs such practises bring. The paper is organised as follows: in Section 2 we sketch the reasons why modellers are interested in solving their models over a long run; in Section 3 we define the algebraic concept of the long run; in Section 4 we specify the key decisions in constructing the extended base on which a model operates and discuss these in relation to an encompassing model in Section 5. Section 6 offers conclusions.[1]

2. The Reasons for Modelling the Long Run

The reasons for specific attention to the long run may be roughly categorized as follows:

(1) Often modellers seek to contribute to the policy debate. Thus models are geared towards examination both of the immediate and long run consequences of alternative scenarios – e.g. whether chosen policy innovations permanently affect economic growth or foreign indebtedness or exchange rate levels etc. Indeed a model which, say, does not incorporate the Government Budget constraint, reaction functions to maintain fiscal solvency, homogeneity restrictions or various identities would generate what might be regarded as fundamentally unsound policy advice. In that respect modellers' attention to long run concerns have made clear the inherent instability of permanent bond finance, the need to finance higher steady state debt via a trade surplus or the inability to target real interest rates etc.

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[1] Throughout this paper we treat the terms 'steady state' and 'long run' fairly synonymously. To illustrate, the long run of the model presented later in the text would involve either collapsing its dynamics (solving for the steady state version) or solving the full dynamic model over an extended simulation horizon (long run); the two methods need not be equivalent. However though model builders may not keep a separate steady-state version they will be aware of and refine its long run properties of their full model; hence the words highlight the same objective.

(2) Examining the long run can often be a good diagnostic device. For example models which are stable and plausible over a “short” horizon may not exhibit such properties over a longer run. Possible instabilities in the model might be drowned out in short run but not so in the long run; this may not necessarily limit their relevance for short-run examination but will invalidate their long run use. For example, Wallis and Whitely (1987) found difficulty solving the steady state version of the City University Business School model which required, amongst other things, changes in the long run deficit financing pattern and a re-modelling of the production function. Similarly, Masson (1987) reports that, when constructing the steady state of MiniMod, finding the ‘correct’ marginal propensity to consume out of wealth was crucial to building a stable steady state version.

Solving and constructing a model’s steady state can illuminate inconsistencies as would be the case if various price/wage/money homogeneity restrictions do not hold. For example, lack of price homogeneity would imply that money has real long-term effects - a proposition that the model-builder might not support or intend.

(3) This concentration on long run issues itself reflects dissatisfaction with older models which typically focused on short run or demand features. Brayton *et al.* (1997) discuss the greater concentration on the supply side of the Federal Reserve’s models after events like the first oil shock and the ‘breakdown’ of the Phillips curve (see also Whitely, 1997). This refocus has prompted more work on theoretical foundations such as microeconomic life cycle features, a greater awareness of policy issues/closures and the inclusion of Rational Expectations to avoid systematic errors in agents’ forecasts etc.

(4) The derivation of a model's long run characteristics facilitates comparison between other models, smaller (or single) equation studies or economic theory in general. This is particularly so if that comparison is over certain key parameters; for example one might expect models to have long run unit elasticities in their money demand-income and their consumption-wealth relationships. Models which generated non-standard results would therefore be forced to explain and rationalise those differences. For example, in deriving the long run of the Bank Of England's small Monetary Model, Currie (1982) comments on the fact that in the long run money demand function, money demand depends positively on long run inflation (rather than negatively as theory implies) and that the demand for public sector debt is inconsistent with a long-run stock/flow equilibrium.

(5) The construction of a model's steady state facilitates the setting of terminal conditions for the full dynamic model. The model is shocked and the resulting long run path of the jump variables is examined. Subsequent simulations can then be set with the 'correct' terminal conditions in place.

The inclusion of Rational Expectations (RE) itself has contributed towards a better understanding and modelling of the long run. RE models tend to advance the effect of long run shocks - since lead variables jump onto the saddle path which thereafter move the model to the equilibrium - and so it is important for RE models to have more sensible and identifiable long run properties than, say, backward looking models. To ensure a unique solution we know that models with forward looking variables - if in linear difference form - should have as many unstable roots (i.e. eigenvalues with roots outside the unit circle) as lead variables (Blanchard and Kahn, 1980). Moreover, a variable's terminal condition should be its steady state solution with the convergence to that steady state governed by the stable root of the system or equation. On a large (highly disaggregated) non-linear model however, it might not be possible to derive the analytical solution for the lead variables and steady state. In this case, arbitrary terminal conditions may be a substitute and the model solved over a sufficiently long horizon that the nature and specification of the terminal conditions do not unduly affect the initial jump in the lead variables.[2]

3. The Concept of the Long Run

Consider the general auto-regressive distributed lag (ADL) equation:

$$A(\mathrm{L})Y_t = B(\mathrm{L})X_t + V_t \tag{1}$$

where A and B are finite polynomials in the lag operator L:

$$A(\mathrm{L}) = 1 - \sum_{i=1}^{I} p_i \mathrm{L}^i \quad \text{and} \quad B(\mathrm{L}) = \sum_{j=0}^{J} \theta_j \mathrm{L}^j .$$

and V_t are well behaved residuals.

Hendry *et al*. (1984) provide a number of testable restrictions on the ADL format to retrieve various economically meaningful relationships such as leading indicators, common factors and

[2] However the popular Fair-Taylor (1983) algorithm provides a way of solving for the 'true' terminal conditions by iteratively extending the simulation horizon. A Type I iterative layer solves the model for fixed expectations terms and a second layer equates the expectations variables and the solution from the first layer. After these layers, the solution period is extended for a set period and solved. If the percentage difference between the latest solution and the previous one within the same solution period is below a prescribed tolerance then this solution procedure (or Type III iteration) is building up the true terminal conditions and solving the model consistently.

error correction mechanisms (ECM) etc. Considering the ECM in itself[3], equation (1) can be rearranged as:

$$A(\mathrm{L})\Delta Y_t = B(\mathrm{L})\Delta X_t - (1-\pi)[Y_{t-q} - \beta_0 - \beta_1 X_{t-q}] + V_t$$

where,

$$A(\mathrm{L}) = 1 - \sum_{i=1}^{p} p_i \mathrm{L}^i \ , \quad B(\mathrm{L}) = \sum_{j=0}^{q} \theta_j \mathrm{L}^j$$

$$\pi = \sum_{i=1}^{q} p_i \ , \quad \beta_0 = 1 \times (1 - \sum_{i=1}^{q} p_i)^{-1} \ , \quad \beta_1 = \sum_{k=0}^{q} \theta_k \times (1 - \sum_{i=1}^{q} p_i)^{-1}$$

which requires

$$(1 - \sum_{i=1}^{q} p_i) \neq 0 \, .$$

The parameter β_1 is the estimate of the long run elasticity between *Y* and *X* (given logarithmic specifications) and will be unity if there is a long run proportionate growth rate between the variables. β_0, incorporating an intercept, is often set to zero.

The **Static State** equilibrium (where $\Delta Y_t = \Delta X_t = 0$)[4] yields (for logarithmic-form models):

$$Y_t = \beta_0 + X_t$$

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[3] The ECM has proved popular since it generates a statistically meaningful regression (ensuring common orders of integration) and also explicitly defines long run relationships between variables and their short run dynamics.

[4] Notice of course that models defined purely in difference terms have no long run solution since the roots lie on the unit circle.

or equivalently, for linear models,

$$Y_t = KX_t$$

where $K = \exp(\beta_0)$. In comparison, the **Steady State** equilibrium (where $\Delta Y_t = \Delta X_t = g$) also yields $Y_t = KX_t$, where

$$K = \exp([B(L) - A(L)]g + (1-\pi)\beta_0)/(1-\pi)$$

Thus, both yield similar solutions, although in the steady state solution the factor of proportionality, *K*, includes terms in the growth rate.

Using the same type of analysis we can examine a full structural model:

$$A(\mathrm{L})Y_t = B(\mathrm{L})X_t + C(\mathrm{L})E_t + V_t$$

where *Y* represents endogenous elements, *X* policy variables, *E* other exogenous factors and *V* a vector of residuals.

From this we can derive the final form:

$$Y_t = [A(\mathrm{L})]^{-1}[B(\mathrm{L})X_t + C(\mathrm{L})E_t + V_t]$$

where the stability of the final form requires that the roots of the polynomial *A*(L) matrix lie within the unit circle. Stacking this yields:

$$Y_t = AY_{t-1} + BX_t + CE_t + V_t$$

or in full matrix form this might become,

$$\begin{bmatrix} Y_1 \\ . \\ . \\ . \\ Y_T \end{bmatrix} = \begin{bmatrix} \pi_2 & 0 & 0 & 0 & 0 \\ \pi_1\pi_2 & . & 0 & 0 & 0 \\ . & . & . & 0 & 0 \\ . & . & . & . & 0 \\ \pi_1^{T-1}\pi_2 & . & . & \pi_1\pi_2 & \pi_2 \end{bmatrix} \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_T \end{bmatrix} + \begin{bmatrix} \pi_3 & 0 & 0 & 0 & 0 \\ \pi_1\pi_3 & . & 0 & 0 & 0 \\ . & . & . & 0 & 0 \\ . & . & . & . & 0 \\ \pi_1^{T-1}\pi_3 & . & . & \pi_1\pi_3 & \pi_3 \end{bmatrix} \begin{bmatrix} E_1 \\ . \\ . \\ . \\ E_T \end{bmatrix}$$

$$+ \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ \pi_1 & . & 0 & 0 & 0 \\ . & . & . & 0 & 0 \\ . & . & . & . & 0 \\ \pi_1^{T-1} & . & . & \pi_1 & I \end{bmatrix} \begin{bmatrix} V_1 \\ . \\ . \\ . \\ V_T \end{bmatrix} + \begin{bmatrix} \pi_1 \\ \pi_1^2 \\ . \\ . \\ \pi_1^T \end{bmatrix} Y_0$$

if from a model whose backward substitution to an arbitrary start yields:

$$Y_t = \pi_1^t Y_0 + \sum_{j=0}^{t-1} \pi_1^j \pi_2 X_{t-j} + \sum_{j=0}^{t-1} \pi_1^j \pi_3 E_{t-j} + \sum_{j=0}^{t-1} \pi_1^j V_{t-j}$$

From this, we can derive the key multiplier relationships:

The Impact Multiplier is: $\frac{\partial Y_t}{\partial X_t} = \pi_2$

The Interim Multiplier is: $\frac{\partial Y_{t+j}}{\partial X_t} = \pi_1^j \pi_2$

The Dynamic or long run multipliers which are the total (finite) impact of a unit step change in a policy variable are:

$$\sum_{\tau=1}^{T} \frac{\partial Y_{i\tau}}{\partial X_{jt}} .$$

For stability purposes, consider again the interim multiplier. As $j \rightarrow \infty$, $\pi_1^j \rightarrow 0$ (of course, if this condition was not satisfied, history would have a cumulatively increasing effect on the present). Thus stability may be redefined as: $\lim_{j \rightarrow \infty} \pi_1^j \pi_2 = 0$, and cumulating this is equivalent to

$$(1 - \pi_1)^{-1} \pi_2 ,$$

this latter form being the final or long run multiplier.

However, most models are non-linear to varying degrees and so yield no unique reduced form since the multipliers are base and perturbation dependent. Checking for the stability, reliability and consistency of models therefore requires forward simulation and the setting up of an extended base which is examined in the next section.

4. Setting Up the Long Run

A number of important decisions have to be made before the long run of a model can be constructed. These may be roughly categorized as in Table 1.

TABLE 1: Long Run Model Decision Map

Section	Sub-Section	Options
Closures	Labour Market	Competitive Keynesian Sticky Wages Real Wage Resistance Exogenous Labour Supply Bargaining Hybrid
	Goods Market	Competitive Imperfect Competition Hybrid
Economic Growth	Population Growth Migration Technical Progress	 Harrod Neutral Hicks Neutral Solow Neutral
Technology	CES Cobb-Douglas Leontief	
Time Horizon	Short-term Long-term	Exogenous Choice General Closures
Policy Rules	Fiscal Monetary	Solvency Rule Balanced Finance Monetary Base Exchange Rate Inflation Targets
Terminal Conditions	General Consumption Exchange Rate	Growth or Differences Trade Balance/Assets

4.1. Closures

The closures for the labour and goods market are particularly important in generating different long run responses to shocks although they will be dealt with briefly here since their specification is entirely model specific. It is clear, however, that models incorporating goods markets with imperfect competition (i.e. P > MC; increasing returns to scale) will behave differently in the long run from competitive models (i.e. P = MC; constant returns to scale technology). For example, if a steady state mark up (over marginal costs) exists then steady state economic activity will be below that for a perfectly competitive economy. The same is true for labour market specifications; for example, fiscal injections have most impact under pure Keynesian closures (such as fixed nominal wages) since they imply an infinitely elastic labour supply passing on all of the demand expansion onto employment. Classical closures however equilibrate wages to their market clearing 'full employment' level whilst other closures such as sticky wages, mark-up, bargaining or real wage resistance have intermediate impacts. In our core model we apply a hybrid model which replicates staggered contracts in the short run.

4.2. Economic Growth

The 'natural' rate of economic growth (g) is equal to the rate of growth of the effective labour force[5] (n) plus the rate of increase in technical progress (α). For example, if the labour force is growing at n and producing α then a full employment (or constant unemployment) equilibrium would require real output growth to equal ($n + \alpha$). All other (domestic) real variables grow at this rate. Since both n and α can be considered exogenous (and subject to off-model calibration) solving for a model's long run real growth rate is relatively straightforward[6]. Indeed there are only three fundamental rates governing the long run of a model: n, α and μ (the rate of growth of the

[5] The labour force growth rate can be disaggregated into domestic and migration-induced components although migration specifications are more often found in General Equilibrium than traditional macro-models.

[6] Policy interventions will not therefore affect the rate of steady state economic growth unless they affect technical progress, population growth and the rate of time preference. Of course, prior to the steady state, policy can affect the (growth) dynamics towards equilibrium. Policy however can affect the steady state level of output mainly from the choice of public debt holdings.

money supply). Thus, to ensure unique (exponential) growth rates for all real and nominal magnitudes we have:

$$(1+g) \equiv (1+n)(1+\alpha) \tag{2}$$

$$(1+\Pi) \equiv (1+g)^{-1}(1+\mu) \tag{3}$$

where Π is core inflation.

Moreover the technical progress element, α, can be modelled as Harrod-, Hicks- or Solow-Neutral. Harrod-Neutral technical progress implies a constant capital to output ratio (hence labour augmenting), Hicks-Neutral (a constant capital-to-labour ratio) and Solow-Neutral is where growth points in the steady state are defined along a constant labour-output ratio (and hence capital augmenting).

Solving for the long run requires post-historical simulation and so the construction of an extended base; this involves forecasts of key variables - e.g. output growth, population, factor prices - as well as policy-mix assumptions made explicit. Subsequently forecasts can be made or inferred for all endogenous variables conditional on assumptions about technical progress and growth in population and the monetary base. The residuals fit the behavioural and identity equations given these assumptions. This represents an extended simulation base - though it does not necessarily imply that the model exhibits well-defined long run properties such as financial neutrality or fiscal solvency since these depend on other factors such as the stability of the model, the specification of the individual equations, policy reaction functions, the level of disaggregation in the price/wage equations and so on[7].

4.3. Technology

The choice of production function is not crucial to the long run of the model since, whatever the choice, steady state output usually coincides with full potential or 'natural rate' output. Production functions tend to be Cobb-Douglas, Leontief, or Constant Elasticity of Substitution (CES):

$$Y = A\left[a(Ne^{\lambda Nt})^{-p} + b(Ke^{\lambda Kt})^{-p}\right]^{-1/p} \tag{4}$$

where: Y is output; A, a, b and p are constants; and λN and λK are labour and capital augmenting technical progress respectively. Invariably, we assume $b = 1 - a$ (constant returns). The elasticity of substitution between capital and labour is given by $\sigma = (1+p)^{-1}$, $\sigma \geq 0$. The special cases $p = 0$ $(\sigma = 1)$ and $p = -1$ $(\sigma = \infty)$ retrieve Cobb Douglas and Leontief forms respectively.

The marginal productivity terms for labour and capital are:

$$\mathrm{MP}_N = A^{-p} a e^{-p\lambda Nt} (Y/N)^{1+p}$$

$$\mathrm{MP}_K = A^{-p} (1-a) e^{-p\lambda Kt} (Y/K)^{1+p}$$

Given perfectly competitive behaviour (which might be expected to hold in the long run) these equate respectively to the real wage and the opportunity cost of capital.

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[7] Long run properties can also be examined with reference to a model's parameterisation - see Deleau *et al*. (1981) and Malgrange (1983). The actual method of solving for the steady state will not be dealt with here but is achieved solving the model with standard iterative techniques with the steady state values as the starting values (Murphy, 1990).

Technical progress can modelled as either embodied or disembodied. In the latter, technology enters as a constant while in the former it is captured by a time trend in the production function (although this often causes problems in generating long run balanced growth in capital and labour see Wallis and Whitely (1987)).

4.4. Policy Assumptions

The methodology on policy rules stems mainly from that of optimal control theory. Given a dynamic linear reduced form,

$$Y_t = \Pi_1 Y_{t-1} + \Pi_2 X_t + \Pi_3 E_t \tag{5}$$

where the explanatory variables are lagged dependent variables, exogenous policy instruments and other exogenous variables, and an additively separate quadratic loss function:

$$L = \tfrac{1}{2}\left[y' \mathrm{Q_Y} y + x' \mathrm{Q_X} x\right] \tag{6}$$

where $y = (Y - Y^{\mathrm{d}})$, $x = (X - X^{\mathrm{d}})$, and Y^{d} and X^{d} are the desired values for targets and instruments respectively. $\mathrm{Q_Y}$ and $\mathrm{Q_X}$ are diagonal penalty cost matrices (cross-variable deviations being usually unpunished) and are, respectively, symmetric positive semi-definite and symmetric positive definite implying that $\mathrm{Q_Y}$ might incorporate some zero penalty costs on target deviations in contrast to $\mathrm{Q_X}$.

Substitution of model (5) into the loss function (6) and differentiating with respect to the instrument set yields an optimal feedback rule of the following general form:

$$X_t^* = \mathrm{F}_t Y_{t-1} + \mathrm{T}_t$$

where, removing time subscripts,

$$F = -\left[Q_X + \Pi_2' Q_Y \Pi_2\right]^{-1}\left[\Pi_1' Q_Y \Pi_2\right]$$

$$T = -\left[Q_X + \Pi_2' Q_Y \Pi_2\right]^{-1}\left[Q_Y \Pi_3 E - Q_X X^{d} - Q_Y \Pi_2 Y^{d}\right]$$

F and T are, respectively, the feedback and feed forward gain. Whilst the feedback gain is time varying but recursive - being related to model parameters and all penalty weights - the feed forward gain is time *in*variant but forward looking - being related to present and future trends in the exogenous and bliss values. Thus policy interventions may be sequentially updated depending on the outturn path for exogenous elements.

A distinction may be drawn between policy types. Open Loop rules - e.g. a fixed money supply growth - involve policies calculated at time t for periods t to $t + i$ ($i > 0$). Alternatively, Closed Loop rules (as illustrated above) take the form of feedback rules sequentially updated in the light of unanticipated shocks and/or changes in the expected outturn for exogenous variables. Of those rules which are of a feedback form we can identify three policy types: proportional, integral and derivative:

Proportional: $$X = \phi(Y^* - Y), \quad \phi > 0 \tag{7}$$

Integral: $$\Delta X = \phi(Y^* - Y), \quad \phi > 0 \tag{8}$$

Derivative: $$X = -\phi \Delta Y, \quad \phi > 0 \tag{9}$$

A proportional policy rule as in (7) links instrument interventions contemporaneously to target failures. However unless $\phi = \infty$ or the rule is supplemented by a term in the steady state instrument value (X^{SS}), target failures - i.e. $(Y^d - Y) \neq 0$ - continue in the steady state. An integral control rule as in (8) relates policy interventions to both contemporaneous and past policy failures and achieves stabilisation with the higher ϕ, the more rapid the convergence. Finally, in a derivative policy rule (9) policy interventions respond purely to the rate of change of the target. Such a rule again is not guaranteed to meet the final target since it is not specified.

The choice of such rules (commonly employed in tax and monetary reaction functions) have a direct bearing on the long run. For example an integral control rule ensures convergence to the target with the speed of convergence to that target given by the feedback behaviour from the rest of the model.

4.5. Monetary Policy

A number of interesting issues arise with monetary policy. For example the long run equilibrium of an economy is invariant to the price level; to remove this 'indeterminacy of the price level' outcome, monetary policy usually ties down the long run price level by, for example, reaction functions from *nominal* interest rates to other *nominal* targets such as inflation, monetary base or bilateral exchange rates. In the latter case inflation will be anchored by the monetary growth rate of the exchange rate hegemon. Similarly it will not be possible to move nominal interest rates to target real rates since it again leaves the price level indeterminate. Moreover leaving nominal interest rates as a policy instrument causes a number of problems in that it is inconsistent with the Uncovered Interest Parity (UIP) relationship and the construction of the yield curve.

4.6. Fiscal Policy

In the long run we would wish fiscal balances to be on a solvent or non-explosive trajectory since otherwise any policy advice derived thereof would not itself prove sustainable[8]. Strictly speaking, solvency implies that the outstanding present debt is less than or equal to the present value of the future expected deficits. More formally, the conventional public accounting identity in continuous time can be written as:

$$S + rb = \dot{b} + \dot{m} \tag{10}$$

where $S = (g + h - t)$ is the primary surplus, g is government expenditures, h is transfers, t is tax revenues, r is the discount rate and $\dot{m}$ is the rate of growth of the monetary base. In discrete time, this can be expressed as:

$$b_t = (1 + r)b_{t-1} + S_t - \Delta m_t \tag{11}$$

i.e.

$$\Delta b_t = S_t + (r - k)b_{t-1} - \Delta m_t \tag{12}$$

and k is the growth rate. If we solve (11) forward in the usual manner (setting $\Delta m_t = 0$ for simplicity), we obtain:

$$b_t = -\sum_{i=0}^{\infty} \mathrm{E}_t \prod_{j=0}^{i} (1 + r_{t-j})^{-1} S_{t+1+i} + \lim_{i\to\infty} \mathrm{E}_t \prod_{j=0}^{i} (1 + r_{t+j})^{-1} b_{t+1+i} \tag{13}$$

where E_t denote expectations of future variables conditioned on the information set available at time t. Thus, we see that that discounted debt must be at least equal to terminal period debt and the discounted sum of (non-interest) balances.

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[8] There is also the question of ensuring a convergent solution generally. Although with terminal conditions elsewhere many models will solve with unsustainable fiscal closures (Smith and Wallis, 1994) but yield no economically meaningful content.

If we transform the above into discounted debt, with a discount factor projected back to the base period:

$$q_i = \prod_{j=0}^{i} (1 + r_j)^{-1}, \qquad q_{-1} = 1$$

we can write (13) in discounted terms:

$$b_t = -\sum_{i=0}^{\infty} \mathrm{E}_t [q_{t+i} / q_{t-1}] S_{t+1+i} + \lim_{i \to \infty} \mathrm{E}_t [q_{t+i} / q_{t-1}] b_{t+1+i} \tag{13'}$$

The terminal (or No Ponzi) condition that we impose on (13') to derive the solvency constraint is that the terminal debt term (or its expectation) goes to zero:

$$\lim_{i \to \infty} \mathrm{E}_t [q_{t+i} / q_{t-1}] b_{t+1+i} \leq 0 \tag{14}$$

and hence

$$b_t = -\sum_{i=0}^{\infty} E_t [q_{t+i} / q_{t-1}] S_{t+1+i} \tag{13''}$$

So solvency implies (13") that the discounted sum of primary deficits equals the initial debt given this terminal condition. With finite horizons, this simply means that public debt in the terminal period is zero; in infinite horizons, the debt must ultimately be serviced either by present and/or future primary surpluses and monetary creation.

Notice two things. First, in (14) we usually discard the inequality sign since we rule out the case of super solvency whereby, in the limit, Governments become net creditors. Second, note that this definition of solvency applies *only* to a dynamically efficient economy. If an economy is dynamically inefficient with growth rates exceeding real interest rates (i.e. $k > r$) in (12), then the debt would forever roll over without the question of solvency arising.[9]

Fiscal solvency therefore requires more than the mere specification of the financing identity. Usually models specify government expenditure and transfers as exogenous whilst tax rules are in (integral control) feedback form and preclude debt explosions which are quite necessary since, in the long run, $(r - k) > 0$[10]. The method of financing fiscal innovations is thus an important element in setting up the long run. Monetary finance implies that monetary expansion accommodates the fiscal one leaving nominal interest rates constant. Under bond finance, monetary policy is not affected and so interest rates rise with increased debt. Invariably in the long run we assume balanced finance (i.e. $\dot{b}^{SS} = \dot{m}^{SS}$) such that the ratio of bonds to money is a constant in the steady state (i.e. portfolio balance).[11]

4.7. Time Horizon

The time horizon cannot be explicitly divided into a short, medium and long run in anything other than a model-specific way but there are qualitative ways in which we can differentiate time horizons. In the short run, certain variables - for example foreign interest rates, world oil prices, population growth, environmental and resource constraints - which might well normally be modelled can be legitimately considered exogenous over a short-term forecasting horizon. If

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[9] Dynamic inefficiency implies that the capital stock is greater than its golden rule level which maximises steady state consumption per capita and so the resource allocation is Pareto sub-optimal. Therefore, the solvency question is predicated on the condition $(r - k) > 0$. Such a condition seems generally consistent with historical data, although there are clearly specific periods for which this condition did not hold. For example, in the 1970s many industrialised countries experienced negative real rates (and hence $k > r$) which made the debt easier to service whilst positive and high real rates ($k < r$) in the 1980s complicated solvency.

[10] This will hold and would also be the case for a permanent bond finance government expansion.

[11] We preclude a permanent increase in bond-financed government expenditures since the debt/gdp ratio would rise without limit requiring an ever increasing build up of foreign liability (as well as positive trade surpluses) which would be incompatible with stock equilibrium. It would also imply an infinite appreciation of the nominal exchange rate given a UIP formulation.

collapsing a larger model into one suitable for forecasting purposes then the user has to isolate those variables to be exogenized. There is some cross-over in this: for example modellers may wish to use long run models but in the short run to exogenize endogenous reactions such as the fiscal solvency rule to examine cases where necessary fiscal adjustments are postponed; see for example Smith and Wallis (1994).

Moreover, models designed for forecasting may not be suitable for policy analysis - for example short run models may not incorporate policy closures rules such as those required to preclude fiscal insolvency, or pay much regard to issues such as long run balanced debt finance of policy, financial neutrality etc. Indeed, there may be a trade-off between a model's theoretical specifications and its forecasting abilities - the implication being that forecasting should be done with small models or cheap time series methods and long run analysis with theoretically well specified (though often highly aggregated) macro-models (Wren-Lewis, 1993)[12]. There is no consensus; whilst many model builders claim their models as purely policy oriented (e.g. Masson *et al.,* 1990), others highlight both their forecasting record and theoretical modernity (e.g. Brayton *et al*., 1997).

Dividing a model up we might say that the short run is characterised by degrees of price and wage *in*flexibility: output is demand determined; incomplete stock adjustment; departures from long run growth; unemployment and output away from full utilisation rates; and the model possibly used for forecasting whilst the long run is characterised by: balanced growth; balanced finance of government debt; price and wage flexibility; flows fully adjusted to stocks; unemployment and

[12] Similarly it is well known that price and wage homogeneity is less likely to hold in highly disaggregated (and hence more forecasting-type) models with the result that it tends to underestimate the monetary transmission mechanism compared to, say, smaller and theoretically tighter models or reduced-forms.

output at ‘normal’ capacity rates; output supply determined; and the model used extensively for policy analysis.

4.8. Terminal Conditions

Terminal conditions tend to be set rather arbitrarily - at a constant value (of the prior period):

$$X^{e}_{t+1} = X_{t} \qquad (15)$$

at a constant growth rate:

$$X^{e}_{t+1} = \dot{X}_{t} / X_{t-1} \qquad (16)$$

or imposed in some way consistent with the priors of the model builder and/or the model’s steady state:

$$X^{e}_{t+1} = X^{\mathrm{SS}} \qquad (17)$$

An example of the latter is often embodied in the treatment of the exchange rate. Despite its limited empirical support (see Messe and Rogoff, 1983) exchange rates are popularly modelled as uncovered interest parity (UIP) meaning the expected appreciation of the dollar exchange rate is set equal to the short-term interest differential in favour of the dollar; this is often modified to include a term in either net foreign assets (NFA) or current account to gdp ratios which proxy a risk premia[13]:

[13] The uncovered interest parity equation caused - at least initially - persistent solution problems since we have a unit root in the forward expectation when we should have a root outside the unit circle to provide saddle-path stability (see, for example, Fisher, 1992). This precludes a unique solution unless the roots of the rest of the model are such as to provide sufficient feedback to obtain an overall solution - although equally endogeneity of either r_t or N_t provides a stable solution to this equation and alters the system root away from unity.

$$E_t = E_{t+1}^e + (r_t - r_t^*) + q\text{NFA}_t, \quad q \geq 0 \tag{18}$$

This equation (being forward looking) still however needs a terminal condition to ensure a unique solution and an arbitrary one like constant growth might be ‘unsatisfactory’ if it implies counter-intuitive movements in the exchange rate and trade variables. The typical constant growth terminal condition therefore is usually supplemented with a term in NFA deviations (from base).

Solving (18) for the first period:

$$E_0 = E_T + \sum_{t=0}^{T}(r_t - r_t^*) + \sum_{t=0}^{T} q\text{NFA}_t$$

This defines the exchange rate’s initial jump defined by its terminal value and the sums of present and future interest rate differentials and net foreign asset ratios. After this initial jump the exchange rate evolves as

$$\Delta E_t = -[(r_t - r_t^*) + q\text{NFA}_t]$$

for a given terminal condition

$$E_{t|t+1} = \theta_1 \dot{E}_t / E_{t-1} + \theta_2 (N_t - N_0)$$

Notice, therefore, that modelling exchange rates as modified uncovered interest parity implies:

(a) The exchange rate jumps in response to any change in exogenous instruments with that change sufficient to clear any effect on net foreign assets brought about by the shock.

(b) The uncovered interest parity formulation implies that monetary policy has no long run output effect since nominal interest rates converge on those of the ‘large’ country - otherwise there would be constant expectations of currency movements which would be inconsistent with a long run *steady state* solution; a permanent interest rate would imply an infinite and hence explosive appreciation.

In this case therefore the choice and specification of terminal condition has a direct bearing on the model's steady state since it produces asset equilibrium in the long run (the NFA ratio stabilising).

5. A Prototype Model

Here we sketch out a small core (annual) macro-model which mirrors the principle elements in a larger one. Its long run is supply determined but the staggered contracts and rational expectations cause disequilibrium and overshooting results in the short run.

Aggregate Demand

S.1 $Y = C + I + G + (X - M)$

S.2 $\Delta c = c_0 + c_1(wealth / c)_{-1} - c_2 RLR + c_3\Delta(y(1-TX))$

S.3 $I = K + (\delta - 1)K_{-1}$

S.4 $\Delta l^d = \mu(y - y^*)$

S.5 $\Delta wages = a_1 + a_2\Pi^e_{t+1} + a_3\Pi_{t-1} + a_4(y - y^*) - a_5(w - p - pr)_{t-1}$

S.6 $WEALTH = \sum_i \varphi_{1i} WEALTH_{t+i} + \varphi_2 Y(1-TX) + K + B + M + \mathrm{NFA}.E$

Aggregate Supply

S.7 $Y = production\ \ function \ \Rightarrow\ MP_L,\ MP_K$

S.8 $\Delta k^d = \zeta((MP_K P_Y)/P_K - UCOC)$

S.9 $UCOC = RL + \delta - \Pi^{e}$

S.10 $P_{K} = zP + (1-z)EP_{IM}$

S.11 $P_{Y} = jP - (1-j)EP_{IM}$

S.12 $P = n(W / MP_{L}) + (1-n)EP_{IM}$

Policy Sector

S.13 $\Delta B + \Delta M = (G - T) + RL_{t-1} \times B_{t-1}$

S.14 $\Delta TX = f(DEBT / GDP, DEFICIT / GDP, \theta_{TX})$

S.15 $\Delta RS = f(MONETARY\ BASE, EXCHANGE\ RATES, Y - Y^{d}, \Pi - \Pi^{d}, \theta_{RS})$

Demand For Liquidity and Overseas Sector

S.16 $CA / Y = \Delta NFA / Y = ((X.PX - IM.PIM) + RS^{**}.\mathrm{NFA}_{t-1}) / Y$

S.17 $E = E_{t+1} + (RS - RS^{**}) + q\mathrm{NFA}$

S.18 $RL / 100 = (\prod_{t=0}^{I-1} (1 + RS_{t+i} / 100)^{1/I}) - 1$

S.19 $m^{d} / p = \beta_{0} + \beta_{1} y + \beta_{2} y_{t-1} + \beta_{3} (m^{d} / p)_{t-1} - \beta_{4} RS - \beta_{5} RS_{t-1}$

S.20 $im = \varepsilon_{1} y - \varepsilon_{2} (ep_{im} / p)$

S.21 $\Delta x = \phi_{1} \Delta m^{**} + \phi_{2} (ep^{**} / p) + \phi_{3} \Delta(e.im.p^{m} / p)$

S.22 $rlr = rl - \Pi^{e}$

Notation: Capital letters symbolise variables in levels and lower-case variables in logarithms; starred (double starred) indicates full capacity (foreign) values; we omit time subscripts except lags and leads. Otherwise obvious notation applies: *RL* and *RS* are the long-term and short-term interest rates respectively; *TX* is the tax rate; l^{d} is labour demand; φ_{2} $(\varphi_{2} < 1)$ is the parameter for labour's share from the production function (φ_{1} incorporates the $(r + \rho + n)$ discount factors); *UCOC* is the user cost of capital; *E* is the exchange rate; *I* = term structure length.

Equation S.1 defines goods market equilibrium. We have already discussed matters relating to the uncovered interest parity formulation (S.17) and production functions (S.7). The tax and nominal interest rate (S.14; S.15) equations are of integral control type and achieve their specified targets. Equations S.10 to S.12 define respectively the investment and value-added deflator and the output price. We omit the 'other' country.

5.1. Consumption

The modelling of consumption reflects the Blanchard (1985) model whereby a single representative consumer maximises expected discounted utility subject to the constraint that the present value of consumption is less than or equal to the initial stock of human and non-human wealth and faces (in the perpetual youth variant) a constant probability of death. Human or labour wealth is simply the present value of disposable income discounted over time by the real equilibrium interest rate (*r*), the (constant) probability of death (ρ) and population growth (*n*):

$$\text{Labour Wealth} = \int_{t=0}^{\infty} Z_t e^{-(r+\rho+n)t} dt \tag{19}$$

where $Z_t = (Y_t - T_t)$.

Given a utility function with constant relative risk aversion, optimal consumption is proportional to wealth by a proportionality factor, α, determined by the three discount factors, the rate of time preference (*tp*) and the degree of relative risk aversion, viz:

$$\text{Consumption} = \alpha(r, \rho, n, tp) \times \text{Total Wealth} \tag{20}$$

where Total Wealth is the sum of labour and asset wealth,

$$\text{Total Wealth} = \text{Labour Wealth} + (K + B + M + \text{NFA})$$

Asset wealth incorporates the capital stock value and holdings of government bonds, high-powered money and net foreign assets. We have therefore incomplete Ricardian Equivalence; human wealth is constrained to cover future tax liabilities, however, since human wealth is discounted at a rate *greater than* the real interest rate (because of positive death probability and population growth rates) the proposition does not fully hold.[14]

Typically in discrete time consumption is modelled as an ECM ensuring that wealth and consumption are homogenous of degree one. In the medium term, consumption is also affected by disposable income (reflecting liquidity constraint considerations) and perhaps other demographic, banking and structural factors embodied in constant or dummy terms. In the long run we see the wealth/consumption ratio is determined by the real interest rate:

$$c = \Psi_0 + wealth - \Psi_1(RLR) \tag{21}$$

where $\Psi_0 = c_0 / c_1$ and $\Psi_1 = c_2 / c_1$. Thus shocks to human wealth only have a transitory effect on consumption since it has a long-run co-integrating relationship with wealth, with their ratio determined by the real interest rate. Hence consumption follows a life-cycle approach in that current income need not necessarily drive current spending decisions.

5.2. Investment

The investment modelling derives from simple classical optimality conditions. The growth of the

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[14] ρ is the constant probability of death and $(1/\rho)$ effectively the horizon index. For $\rho > 0$ $(\rho = 0)$, we have finite (infinite) horizons for consumers. Ricardian Equivalence holds for $\rho = 0$ since consumers will live long enough to meet the implied future increase in taxes from previous debt issues.

capital stock depends on the difference between the marginal product of capital (obtained from differentiating the production function) and the real user cost of capital (UCOC). The steady state UCOC is a function of the long run interest rate and the rate of deprecation.

5.3. Money Demand and Supply

Money Demand comes from the quantity-theory identity $M = aPY$ where a is the inverse of the velocity of money and represents its opportunity cost typically proxied by interest rates:

$$M^{d} = PYe^{-bRS} \tag{22}$$

Typically money demand functions are of the following ADL form[15]:

$$(m^{d} / p)_{t} = \beta_0 + \beta_1 y_t + \beta_2 y_{t-1} + \beta_3 (m^{d} / p)_{t-1} - \beta_4 RS_t - \beta_5 RS_{t-1}$$

The Static long run solution is:

$$(m^{d} / p)_{t}^{\mathrm{SS}} = (1-\beta_3)^{-1} \times [\beta_0 + (\beta_1 + \beta_2) y_t^{\mathrm{SS}} - (\beta_4 + \beta_5) RS_t^{\mathrm{SS}}] = (m^{s} / p)^{\mathrm{SS}}$$

In the steady state, therefore, Money Demand equals Money Supply, output reverts to its natural rate and interest rates equate money demand and supply and fulfil the Uncovered interest parity equation. The long-run demand for real balances therefore is invariant to the inflation rate. Moreover the equilibrium price level can be solved as:

$$P^{*} = (1-\beta_3) M^{s*} \times [\beta_0 + (\beta_1 + \beta_2) Y_t - (\beta_4 + \beta_5) RS_t]^{-1} \tag{23}$$

---

[15] Variables are in natural logarithms except nominal interest rates since that would impose an unrealistic constant elasticity. Price homogeneity is imposed as above in order that the demand for money becomes the demand for *real* money balances. Interest rates are rationalised as the (opportunity) cost of holding real money balances but if the interest rate is a policy variable this equation may be reformulated with inflation acting as a substitute or supplement to interest rates.

If $(1-\beta_3)^{-1} \times (\beta_1 + \beta_2) = 1$, then this implies that real money demand was homogenous of degree one in real income - i.e. financial neutrality. Solving for the steady state equilibrium we differentiate (23) with respect to time and (given zero long run growth in the rate variables) we retrieve equation (3).

Short run nominal interest rates form part of an integral control rule around some nominal target but in the long term are determined by the UIP equation (and hence by the 'Large Country' monetary policy).We have a conventional term structure for long run nominal rates (S.18). In the steady state long run nominal rates converge on short rates after a lag determined by the length of the term structure:

$$rl^{\mathrm{SS}} = rs^{\mathrm{SS}} \tag{24}$$

The real interest rate (*rlr*) is given by (S.22) - in the long run Π^e converges on core or steady state Π, essentially derived out as in (3).[16] Essentially therefore the *rlr* is exogenous since it depends on nominal rates - set by large country monetary policy - and monetary base and economic growth both exogenously determined.

5.4. Labour Markets

We see that wage adjustments are sluggish on over-lapping contracts reasoning.The specification for wages implies imperfect adjustment to labour market equilibrium and expected inflation.

$$\Delta wages = a_1 + a_2\Pi^e_{t+1} + a_3\Pi_{t-1} + a_4(y - y^*) - a_5(w - p - pr)_{t-1} \tag{24}$$

---

[16] Inflationary expectations may be set in a model consistent manner or as some weighted sum of backward and consistent components.

where Π is inflation rate in consumption prices, *PR* a long run productivity trend and Y^* is full capacity output. Thus wages are a function of labour market disequilibrium, the real product wage and are assumed to adjust imperfectly to inflation. The interpretation of these parameters is straightforward: reducing a_1, a_2, a_4 or a_5 would increase market sensitivity by increasing market responsiveness to demand conditions, labour market disequilibria or conditions in the labour market itself. A fall in a_2 (a_4) implies greater real (nominal) wage rigidity.

If $a_2 = (1 - a_3)$ - as they would be if expectations were a weighted average of rational and backward looking components - then we could cancel the wage and price inflation terms which would be growing at the same rate. If output reverted to its 'natural' rate then we would expect the real product wage to equal the productivity terms and whatever structural factors are embodied in the intercept.

5.5. Trade Variables

The Current Account equation (S.16) is of particular interest; if in the steady state $\Delta\text{NFA} = g$ then (S.16) can be re-expressed as:

$$((X_t PX_t - M_t PIM_t) + r_t^{**}\text{NFA}_{t-1}) / Y_t = (\text{NFA}/Y).((g - r)/(1+g)) \qquad (25)$$

This implies that steady state debtor countries (e.g. $\text{NFA}^{\text{SS}}/Y^{\text{SS}} < 0$) must run a positive trade surplus and *vice versa*; given that $g^{\text{SS}} - r^{\text{SS}} < 0$ and the stock of NFA to income is a constant.[17] This has implications for fiscal policy; for example a permanent debt/income expansion increases consumption in the short run (through normal Keynesian channels) but - if financed by increases

[17] This also depends on how and if the terminal condition on the exchange rate handles net foreign assets.

in foreign indebtedness - implies *lower* steady state consumption as the trade balance moves inevitably into surplus.[18]

Imports depend on domestic output and the relative price of domestic and import prices - the equilibrium of which is simply the replacement of the steady state value for each variable in the equation. Exports react to foreign imports, the gap between foreign and domestic prices and the change in the deflated value of imports. In the long run we would expect exports and imports to grow at a common rate (for $\phi_1 = 1$) and the price ratios to be constant in the steady state - the long run equilibrium for export demand ensures that imports respond to the foreign/domestic price ratio, a constant in the steady state.

6. Conclusions

This paper has attempted to briefly survey and motivate the incorporation of long run elements into macro economic models. We have suggested that, *inter alia*, modellers are interested in the long run for reasons of theory (for example, to ensure sustainable policy closures and tighter theoretical foundations) and also for algorithmic convenience (for example, in setting and resolving appropriate terminal conditions). We have ignored many related issues such as cointegration analysis in macro models and numerical issues in solving for the steady state etc but have suggested other more dominant themes common to supply-driven macro models.

These themes may be roughly listed as:

[18] This also implies a steady state depreciation of the exchange rate to produce the trade surplus.

- **Balanced Growth**
- **Balanced Public Finance**
- **Homogeneity Restrictions**

  For example in prices, wages, money demand, constant returns technology etc.
- **Money Neutrality**
- **Sustainable Policy Feedback Rules**

  This implies not only that their parameterisation leads to unstable feedbacks but also that base fiscal projections are internally consistent.
- **Long Run Vertical Phillips Curves**

Construction of a steady state model from a larger dynamic one yields several obvious benefits in terms of being numerically more straightforward to simulate as well as forcing model builders to consider their overall model structure and its theoretical coherence. Indeed an “appropriate” long run specification is crucial to understand the full policy and stock-flow implications of certain permanent shocks and this is where such improvements have enriched our analysis over earlier mainly demand-driven models or simple text book flow tools like the Mundell-Flenming IS-LM-BP framework.

The trade-off that such practises bring might be that models with a large emphasis on theoretical and long run coherence may have a poor forecasting record. This is often of course to the immediate financial disadvantage of private-sector modelling groups who depend on the commercial saleability of their model. It could well be argued however that forecasting can be done relatively cheaply with small reduced form models or time series approaches leaving policy analysis in the hands of models with some explicit theoretical long run foundation.[19]

[19] Fisher and Whitely (1997), for example, look at the different models that the Bank of England uses for policy analysis.

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