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STRATEGIC DEFAULT IN FINANCIAL NETWORKS

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Abstract. This paper investigates a model of default in financial networks where the decision by one agent on whether or not to default impacts the incentives of other agents to escape default. Agents' payoffs are determined by the clearing mechanism introduced in the seminal contribution of Eisenberg and Noe (2001). We first show the existence of a Nash equilibrium of this default game. Furthermore, we develop an algorithm to find all Nash equilibria that relies on the financial network structure. The algorithm provides a ranking for the set of Nash equilibria, which can serve as a measure of systemic risk. Finally, we show that introducing a central clearing counterparty achieves the efficient equilibrium at no additional cost.

JEL classification: C72, D53, D85, G21, G28, G33.

Keywords: systemic risk, default, financial networks, coordination games, central clearing counterparty, financial regulation.

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1. Introduction

Financial institutions carry out various transactions with each other, including risk–sharing and insurance. The architecture of the network of transactions between institutions can support financial stability because it enables them to share funding or transfer risk. But these linkages can also facilitate the diffusion of shocks through the system, due to chains of default and the domino effect. This is referred to as systemic risk. Systemic risk is costly for individuals, institutions and economies, as demonstrated by the last financial crisis. The obvious need for a stable financial system has led to a significant interest in policies that could reduce systemic risk and mitigate contagion.

This paper introduces a model of default in financial networks. We study a two-period economy where agents have a positive endowment in each period. The endowment represents agents' cash flows from outside the financial system. We assume that agents hold each other's financial liabilities and that this constitutes the network between them. These liabilities mature in the second period, and we assume that agents' second-period endowments are small and deterministic, so that they face a risk of default. More specifically, the liabilities structure results in cyclical payments interdependencies that are simultaneously computed according to the clearing mechanism described in the seminal contribution of Eisenberg and Noe (2001). The clearing vector satisfies three criteria:

- debt absolute priority, which stipulates that liabilities are paid in full in order to have positive equity;
- limited liability, which means that the payment made by each agent cannot exceed its inflows;
- equal seniority of all creditors, which implies pro rata repayments.

Agents can avoid default by storing part of their first-period endowment.

Due to complementarities in the payments, the decision taken by one agent to store part of his endowment exerts a positive externality on the other agents to whom he is connected.[1] We show that the strategic interactions in the financial system modelled here can be investigated as a coordination game, called the default game, where agents' decisions are simply whether to default or not. It is well known in the literature that

[1]The non-storage in our model can be equivalently interpreted as a bank run in the influential Diamond–Dybvig model.

coordination games will in general yield multiple pure–strategy Nash equilibria and that the set of pure–strategy Nash equilibria has a lattice structure—in particular, there are two extreme pure–strategy Nash equilibria. In our setting, the best equilibrium is the one where the largest number of agents choose the maximal action Non-Default and the worst equilibrium is the one where the largest number of agents choose the minimal action Default.

In the paper, we relate the multiplicity of Nash equilibria to the presence of a cycle of financial obligations. Then, we develop a simple algorithm for finding all Nash equilibria of the default game. While there are easy algorithms for finding the maximal and minimal equilibria and relatively easy algorithms to compute all Nash equilibria in coordination games such as the default game (see Echenique (2007)), the advantage of the algorithm developed in this paper is that it relies on the financial network structure to inform the computation of Nash equilibria. Algorithms that exploit the network structure such as the algorithm developed in this paper, as well as quickly computing all Nash equilibria, provide useful information on the strategic interactions between agents. In particular, the algorithm provides a ranking for the Nash equilibria in each strongly connected component of the financial network. The ranking for the Nash equilibria is advantageous from a policy perspective since it can serve as a measure of systemic risk contribution of agents, More specifically, agents that default in all Nash equilibria will be called the *first wave of default*. Then, agents that default in all Nash equilibria except the highest Nash equilibrium will be called the *second wave of default* and so on.

In this paper, we show that the problem of inefficient coordination may arise in financial networks. Similar to other areas in economics, the strategic complementarities of payments due to the cyclical financial interconnections allow for the existence of multiple Nash equilibria. This gives rise to the question of which one of these equilibria will be the outcome of the underlying default game. From a policy perspective, given that inefficient coordination might pose a severe economic problem, there is a need for financial institutions fostering efficient coordination of agents' decisions. Recently, central clearing has become the cornerstone of policy reform in financial markets since it limits the scope of default contagion. Our analysis shows that introducing a central clearing counterparty (henceforth, CCP) also allows agents playing different actions at different Nash equilibria to coordinate on the efficient equilibrium at no additional cost. As a consequence, our result reinforces the key role CCP's play in stabilising financial markets.

This paper is structured as follows. In Section 2, we go over the related literature. Then we describe the model and show the existence of a Nash equilibrium in Section 3. We develop an algorithm to find all Nash equilibria in Section 4 and Section 5 provides some policy implications of central clearing. Section 6 concludes the paper and Section 7 is an appendix devoted to the proofs.

2. Related Literature

The impact of the financial network structure on economic stability has been a subject of ongoing interest since the last financial crisis (of 2008). The seminal contributions of Allen and Gale (2000) and Eisenberg and Noe (2001) were first to acknowledge that the financial network structure determines default contagion, and would serve as a basis for many subsequent contributions.

Allen and Gale (2000) investigate how symmetric financial networks lead to contagion, where links represent sharing agreements. Their key finding is that incomplete financial networks are less resilient and more vulnerable to contagion than their complete counterparts. Eisenberg and Noe (2001) develop a static model of default contagion in a financial network where agents hold each other's financial liabilities and the activities and operations of each agent are condensed into one value: the operational cash flow. The repayment of liabilities will be interdependent, since whether an agent defaults or not is a result of his operational cash flow as well as the payments he receives from other agents. Eisenberg and Noe first prove the existence of a clearing payment vector that is unique under mild conditions. They also provide an algorithm to compute the clearing vector, which is important to predict chains of defaults.

Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) extend the Eisenberg–Noe model to accommodate agent exposure to outside shocks. They establish that up to a certain magnitude of shocks, the more connected the financial network is, the more stable it is; beyond this threshold, the connectedness of the network makes it more prone to contagion and thus more fragile. Elliott, Golub and Jackson (2014) introduce two concepts of cross-holdings that have distinctive and non-monotonic impact on default cascades.

Integration, which measures the dependence on counterparties, expands the extent of default contagion but reduces the probability of the first failure; while diversification, which measures the heterogeneity of cross-holdings, increases the propagation of failure cascades but decreases the exposure level among pairs of financial institutions. Cabrales, Gottardi and Vega-Redondo (2017) investigate the optimal network structure that maximizes risk-sharing benefits among interconnected firms while decreasing their risk exposure. Jackson and Pernoud (2020) investigate how the network structure impacts agents' investment strategies as well as optimal regulatory intervention. Other recent contributions include Teteryatnikova (2014) and Csóka and Herings (2016).

For a recent survey, see Jackson and Pernoud (2020). Several approaches have been investigated to mitigate the domino effect in the financial network, such as central clearing and identifying the most systemically relevant financial institutions and then targeting them through cash injections. For instance, Demange (2018), following a similar approach to Eisenberg and Noe (2001), develops a new measure, called the *threat index*, which identifies the most systemically relevant agents for optimal targeted cash injection.

3. The Model

Consider a two-period ($t = 1, 2$) economy with $N = \{1, 2, ..., n\}$ agents. Agent i's endowment in the first period is $z_i^1 \geq 0$ and in the second period is $z_i^2 > 0$. The endowment of agent i in each period denotes the cash flows arriving from outside the financial system. We assume that agents hold each other's liabilities, which mature in the second period. More specifically, given two agents $i, j \in N$, let $L_{ij} \in \mathbb{R}^+$ denote the liability that agent i owes agent j. Then, agent i's total liabilities are $L_i = \sum_{j \in N} L_{ij}$. Meanwhile, $\sum_{j \in N} L_{ji}$ is the total assets of agent i. Let $\boldsymbol{\alpha} = (\alpha_{ij})_{i,j \in N}$ denote the matrix of relative liabilities, with entries $\alpha_{ij} = \frac{L_{ij}}{L_i}$ representing the ratio of the liability agent i owes to agent j over the total amount of agent i's liabilities.

Each agent i can store an amount $x_i \in [0, z_i^1]$ from his first-period endowment and receives an interest rate $r > 0$ on his storage. Given the storage strategies of agents $\mathbf{x} = (x_i)_{i \in N}$, let $\boldsymbol{\pi}^{\mathbf{x}} = (\pi_i^{\mathbf{x}})_{i \in N}$ denote the clearing payment vector, uniquely defined as in

Eisenberg and Noe (2001), such that for each agent i it holds that

$$\pi_i^{\mathbf{x}} = \min\left\{ z_i^2 + (1+r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}}; L_i \right\}.$$

This means that $z_i^1 - x_i$ denotes the equity of agent i in the first period and

$$z_i^2 + (1+r) x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}$$

denotes the equity of agent i in the second period.

The utility function of agent i is $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$, where e_i^1 is the equity of agent i at $t = 1$ and e_i^2 is the equity of agent i at $t = 2$. Therefore, the utility function of agent i, given the storage strategies of agents $\mathbf{x} = (x_i, x_{-i})$, is

$$U_i(z_i^1 - x_i, z_i^2 + (1+r)\, x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}) = z_i^1 + z_i^2 + r x_i + \sum_{j=1}^{n} \alpha_{ji} \pi_j^{\mathbf{x}} - \pi_i^{\mathbf{x}}.$$

4. Nash Equilibria of the Default Game

First, we investigate further the economy introduced above. Observe that each agent will choose to store a positive amount of his first-period endowment if and only if he prefers (is better off) not to default; otherwise he will store nothing. If he prefers not to default, since his utility is linear and the interest rate $r > 0$ he will store his entire first-period endowment. Similarly, it is only the decision of an agent to default or not, rather than the amount of storage, that affects the other agents. This is because, if he defaults, he will pay out his total second-period equity and, if he does not default, he will pay his total liability, neither of which is directly affected by his level of storage.

Therefore, the strategic interaction of agents in the economy can be investigated as a binary coordination game with two actions (Default) = 0 and (Non-Default) = 1 among which agents must choose. Now, define a threshold $\tau_i(\mathbf{a}_{-i})$ as the minimum amount agent i must pay in the second period to avoid default, given other agents' actions $\mathbf{a}_{-i}$.

Proposition 1. *The threshold $\tau_i(\mathbf{a}_{-i})$ is well-defined and decreasing in $\mathbf{a}_{-i}$.*

Proof. The proof of Proposition 1, together with all our other proofs, appears in the Appendix.□

Proposition 1 shows that the threshold $\tau_i(\mathbf{a}_{-i})$ is well-defined. Observe that agent i will choose to play 1 whenever

$$(1+r)z_i^1 - \tau_i(\mathbf{a}_{-i}) \geq z_i^1.$$

Therefore, the best reply function of agent i can be written as follows:

$$\Psi_i(\mathbf{a_{-i}}) = \begin{cases} 1 & \text{if } rz_i^1 - \tau_i(\mathbf{a}_{-i}) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

A profile of actions $\mathbf{a}^* \in \{0,1\}^N$ is a Nash equilibrium if $a_i^* = \Psi_i\left(\mathbf{a}^*_{-\mathbf{i}}\right)$.

The default game introduced above corresponds to a binary game of strategic complements. As defined in Topkis (1979), Milgrom and Roberts (1990), and Vives (1990) strategic complementarities arise if an increase in one agent's strategy increases the optimal strategies of the other agents.[2]

Theorem 1. *There exists a pure–strategy Nash equilibrium of the default game.*

Theorem 1 shows the existence of a pure–strategy Nash equilibrium. Understandably, the existence of a pure–strategy Nash equilibrium follows from the strategic complementarities between agents' actions, since the decision of an agent not to default makes it easier for other agents not to default too.

It is established in the literature that a binary game of strategic complements will in general have multiple pure–strategy Nash equilibria with a lattice structure. In particular, this class of games has two extreme equilibria: the best equilibrium is the equilibrium where the largest number of agents choose the maximal action (Non-Default) $= 1$ and the worst equilibrium is the equilibrium where the largest number of agents choose the minimal action (Default) $= 0$.

For simplicity, for the remainder of this paper, we assume that at a Nash equilibrium of the default game, no agent is indifferent between (Non-Default) $= 1$ and (Default) $= 0$,

[2]See, Bulow, Geanakoplos and Klemperer (1985), Sobel (1988), Echenique and Sabarwal (2003), Amir (2005), Echenique (2007) and Barraquer (2013) for other economic applications of games of strategic complements.

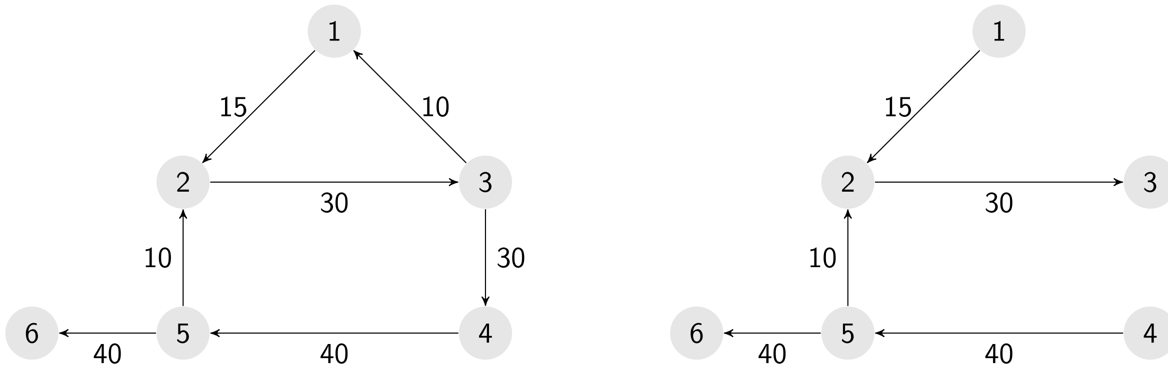

Figure 1. Cyclical obligations Unidirectional obligations

which is likely to be the case.[3] The following result highlights the connection between the multiplicity of equilibria and the structure of the financial network.

Proposition 2. *If the default game has multiple Nash equilibria then, the financial network has cyclical obligations.*

Proposition 2 shows that the presence of a cycle of financial obligations is generically necessary for the multiplicity of Nash equilibria. Eisenberg and Noe (2001) term this phenomenon cyclical interdependence and illustrate it as follows: "*A default by Firm A on its obligations to Firm B may lead B to default on its obligations to C. A default by C may, in turn have a feedback effect on A.*"

In the following, we will show that the close relationship between the multiplicity of Nash equilibria and the cyclical financial interconnections is useful to solve for pure–strategy Nash equilibria of the default game. More specifically, we will provide an algorithm to find all pure–strategy Nash equilibria of the default game.

Recall that the financial network is strongly connected if there is a path of obligations between all pairs of agents. A strongly connected component (henceforth, SCC) of the financial network is a maximal[4] strongly connected subnetwork.

4.1. **A financial network with a unique SCC.** First, for simplicity, we consider the case of a financial network with a unique strongly connected component. We will use the following notion of *ear decomposition* of a network, which is useful given its close

[3]That is, this always holds except for a finite set of first-period endowments.

[4]In the sense that it is not properly contained in a larger SCC.

relationship to network connectivity. An ear decomposition of a network is a partition of the set of agents into an ordered collection of agent-disjoint simple paths, called ears. More precisely, an ear decomposition of a network is a partition of the agents into $E_0, E_1, \ldots, E_p$ such that

- E_0 is a cycle
- for each $h = 1, \ldots, p$, it holds that $E_h = \{v_{1_h}, \cdots, v_{k_h}\}$ is a directed path such that the start agent v_{1_h} and the end agent v_{k_h} of each E_h are in $E_1 \cup \ldots \cup E_{h-1}$ but the internal agents of E_h—that is, $v_{2_h}, \cdots, v_{(k-1)_h}$—are not in $E_1 \cup \ldots \cup E_{h-1}$.[5]

A financial network is strongly connected if and only if it has an ear decomposition. In the following, we will rely on the ear decomposition to provide an algorithm to find all pure–strategy Nash equilibria of the default game of a financial network with a unique SCC.

Given an ear E_h, we define an *injection agent* of the ear as an agent (on E_h) with inflows from one or more other ears. We define the *activation level* λ_i of an agent i as the minimal inflow that will allow the agent to escape default. The *activation outflow* $A_j(v_i)$ is the outflow of node j that is sufficient for agent v_i to escape default.

An injection agent will essentially have more than one debtor illustrated by several ingoing links and will comprise uncertainty at the level of the payment needed for it to escape default. The algorithm, which we call USCCNE, builds on the above definitions and goes as follows:

Sketch of Algorithm 1. (USCCNE) *A complete description can be found in the Appendix.*

The algorithm traverses the network following the structure of the ear decomposition, starting from the final ear E_h and working backwards to E_1. At each ear E_k, the algorithm traverses the network from v_{k_1} outwards using breadth-first search, recursively calculating the activation outflows from v_{k_1} for each node in the search tree.

When arriving at an injection node, the algorithm needs to take account of feedback. If the other incoming edges of the injection node are outgoing edges from visited ears, then the activation outflow can be calculated recursively from the quantities already calculated.

[5] Each E_h ($h = 1, \ldots, p$) is called an ear.

If the other incoming edges are from unvisited ears, then the breadth-first search stops and the algorithm moves to the next ear.

The algorithm makes the search for equilibria a recursive problem. The algorithm works backwards through the ear decomposition of the graph, but, starting from a given ear, works forwards through the network until it reaches a node whose feedback information hasn't yet been calculated.

Proposition 3. *(i) USCCNE identifies all equilibria.(ii) USCCNE returns only equilibria.*

Proposition 4. *USCCNE has a worst-case time complexity of* $\mathcal{O}(n^4)$.

USCCNE is particularly fast when there are fewer edges (liabilities) in the default game, and as a result, fewer ears. The number of ears in the network is equal to $|E| = m - n + 1$, where m is the number of edges. When the network has fewer, longer ears, the algorithm traverses the network more quickly, visiting individual agents less frequently. For example, given a cycle network, traversal of the network and calculation of consistent strategy profiles is completed in linear time.

The key feature of the USCCNE is that it transforms the SCC into a tree-like structure by considering the possible payments of the start node of the second ear. It also ranks the different agents by their order of non-default according to their activation level.

Corollary 1. *The USCCNE algorithm provides as well a ranking for the set of Nash equilibria.*

Existing literature indicates that the set of pure strategy Nash equilibria of a coordination game has a lattice structure. Corollary 1 shows that the USCCNE algorithm developed in this paper based on the concept of ear decomposition provides the stronger property of ranking Nash equilibria within the SCC.[6]

In interpretation, the ranking of Nash equilibria could be thought of as a measure of systemic risk based on waves of default. That is, the agents that default in all Nash equilibria will be called the *first wave of default.* Then, agents that default in all Nash equilibria except the highest Nash equilibrium will be called the *second wave of default.*

[6]More specifically, given two Nash equilibria x and x' of the default game, it holds that $\inf\{x, x'\} = \min\{x, x'\}$ and $\sup\{x, x'\} = \max\{x, x'\}$.

Then, agents that default in all Nash equilibria except the two highest Nash equilibria will be called the *third wave of default.* and so on.

The next example illustrates the default game.

Example 1. Consider an economy of ten agents connected through their ownership of each other's liabilities, among which only the first nine agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (25, 25, 40, 35, 40, 40, 25, 70, 20)$ and in the second period are $\mathbf{z}^2 = (3, 3, 3, 3, 3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the financial network in Figure 2.

This financial network contains a unique SCC, $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, which has three ears, $E_0 = \{1, 2, 3, 4, 5, 1\}; E_1 = \{3, 6, 7, 8, 2\}$; and $E_2 = \{7, 9, 1\}$.

In order to compute the Nash equilibria, we apply USCCNE. Figure 3 shows the trees generated by the algorithm. Activation outflows from agents 3 and 7 are calculated recursively. For example, the activation inflow to agent 4 is $\lambda_4 = L_4 - z_4^2 - 0.1z_4^1 = 8.5$. Thus, agent 4 must receive a minimum of 8.5 by 3 to escape default, which corresponds to a minimum payment by 3 of $A_3(v_4) = 42.5$. Activation outflows $A_3(v_5), A_3(v_1), ...$ are calculated recursively, with agent $A_3(v_5)$ dependent on $A_3(v_4)$ without loss of generality.

For each tree, activation levels are sorted and the algorithm cycles through the sorted activation levels in order to identify strategy profiles that are consistent taking into account only the strategies of other agents within the tree. For tree T_7, the tree-consistent strategy profiles $\Gamma^*(T_7)$ are $\gamma(v_7, v_8, v_9) \in \{(0, 0, 0), (1, 1, 1)\}$, where a strategy of 0 refers to default, and 1 refers to non-default.[7] For tree T_3, the sorted activation levels yield the following tree-consistent strategy profiles: $\Gamma^*(T_3)$ are $\gamma \in \{(0, 0, 0, 0, 0, 0, 0, 0, 0),$ $(0, 0, 0, 1, 1, 0, 0, 0, 0), (1, 1, 0, 1, 1, 0, 1, 1, 1), (1, 1, 0, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1)\}$.

We verify each of the above strategy profiles above, eliminating the strategy profiles $(1, 1, 0, 1, 1, 0, 1, 1, 1)$, and $(1, 1, 0, 1, 1, 1, 1, 1, 1)$, where the decision of agent v_3 to default is not a best response to the inflows received. Ultimately, the algorithm yields

[7]Where an individual tree has more than two tree-consistent strategy profiles, then the tree will have multiple activation levels where it is a child in subsequent trees.

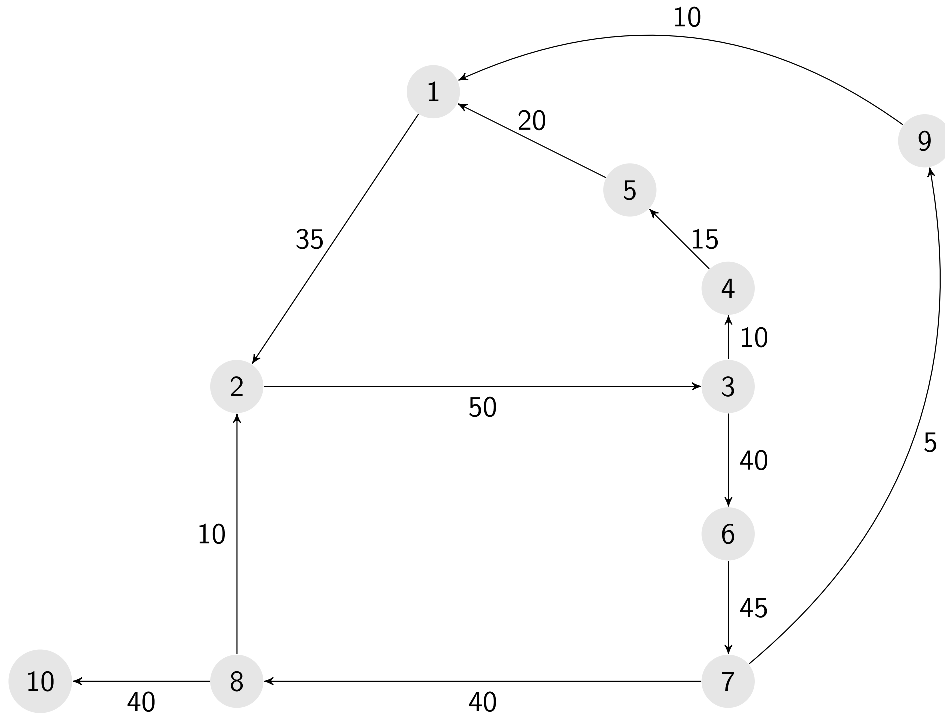

Figure 2. A financial network with ten agents

the following equilibrium strategy profiles: $\{(0,0,0,0,0,0,0,0,0,0),\ (0,0,0,1,1,0,0,0,0,0),$ $(1,1,1,1,1,1,1,1,1,1)\}$.

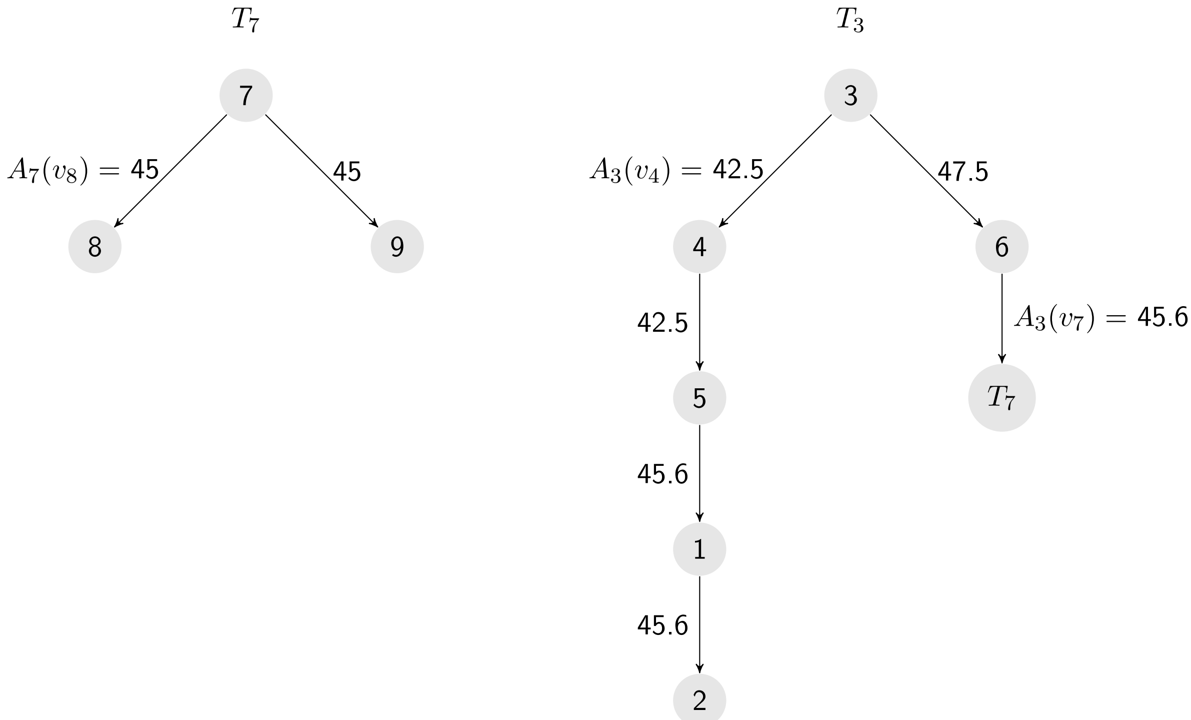

Figure 3. The tree decomposition generated by the algorithm. Edge labels indicate the outflow from T_k that activates the destination node of the edge.

4.2. **Arbitrary financial network.** Now we investigate the case of an arbitrary financial network. Recall that an arbitrary financial network can be transformed into a *directed acyclic graph* (henceforth, DAG)—that is, a network with no cycles–by contracting each SCC into a single large node (see Figures 6-7).

The algorithm described here (MSCCNE) is a generalisation of USCCNE. It consists of applying the USCCNE to each SCC in any given arbitrary network starting by the SCCs with no incoming link from any outside node or group of nodes, which are the SCCs that are not impacted by the other nodes in the network, and moving along the chain of SCCs.

In the following, we will rely on *transitive reduction*, which is a uniquely defined operation on a DAG, to compute the pure–strategy Nash equilibria of a financial network with multiple SCCs. A transitive reduction of a DAG is the network representation with the fewest possible links that preserves the chains of default of the original financial network(see Figure 8). It is hence constructed by removing all the links that are unnecessary for the chain of default to be realised and only the nodes which were connected by a

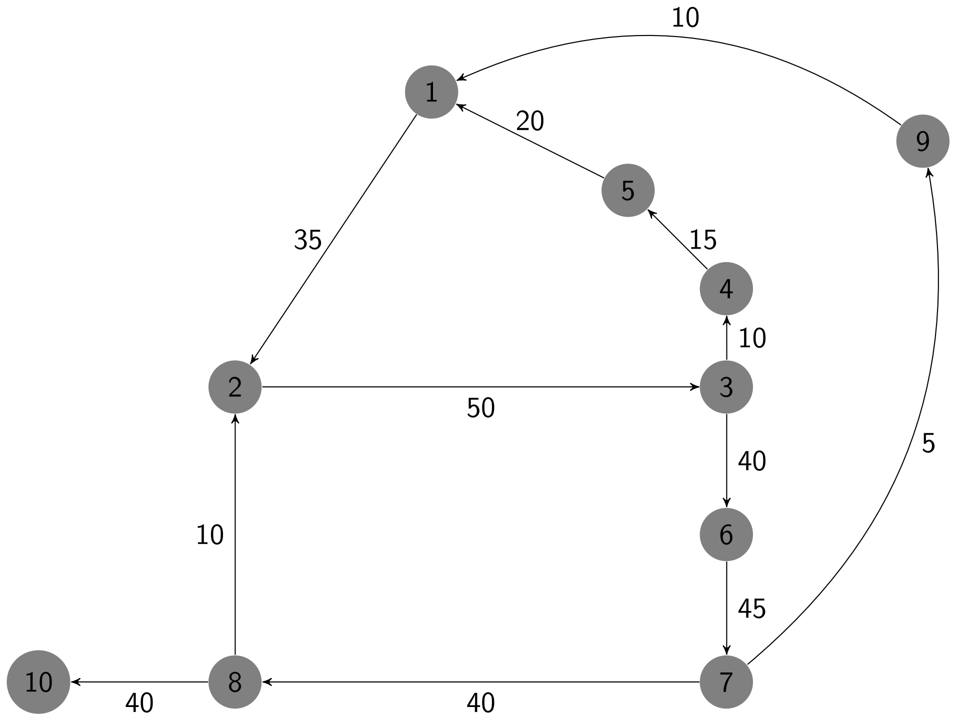

Figure 4. The best equilibrium

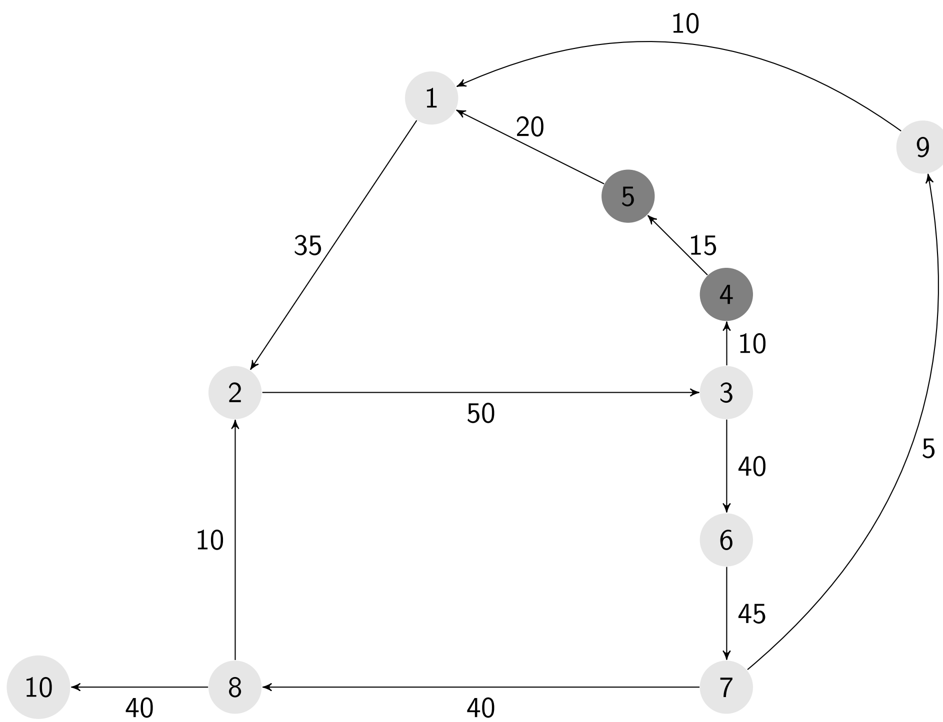

Figure 5. The intermediate equilibrium

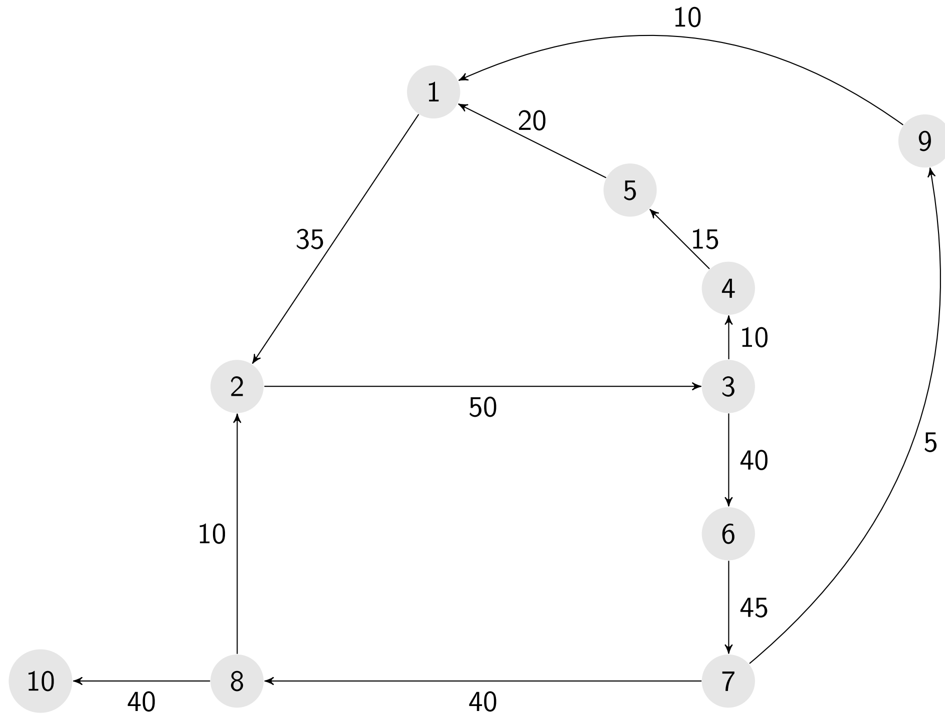

Figure 6. The worst equilibrium

path in the original network remain connected in the transitively reduced network. For instance, if A links to B, and B links to C, then the transitive reduction removes the link from A to C, if it exists.

Observe that, from the minimality of links in the transitive reduction, there exists a unique partition of the set of agents $\mathcal{W} = \{W_1, \dots, W_k\}$ such that W_1 corresponds to the SCCs with no incoming links, W_2 corresponds to the SCCs with only incoming links from W_1, W_3 corresponds to the SCCs with only incoming links from $W_1 \cup W_2$, and so on.

Then, the algorithm USCCNE can be easily extended to compute the Nash equilibria with multiple SCCs. The algorithm, which we call MSCCNE, goes as follows:

(1) Apply USCCNE to find all Nash equilibria for each SCC in W_1.
(2) For each product of Nash equilibria of SCCs in W_1, apply USCCNE to find all Nash equilibria for each SCC in W_2.
(3) For each product of Nash equilibria of SCCs in $W_1 \cup W_2$, apply USCCNE to find all Nash equilibria for each SCC in W_3.
(4) Repeat the procedure until visiting all the elements of the partition $\mathcal{W}$.

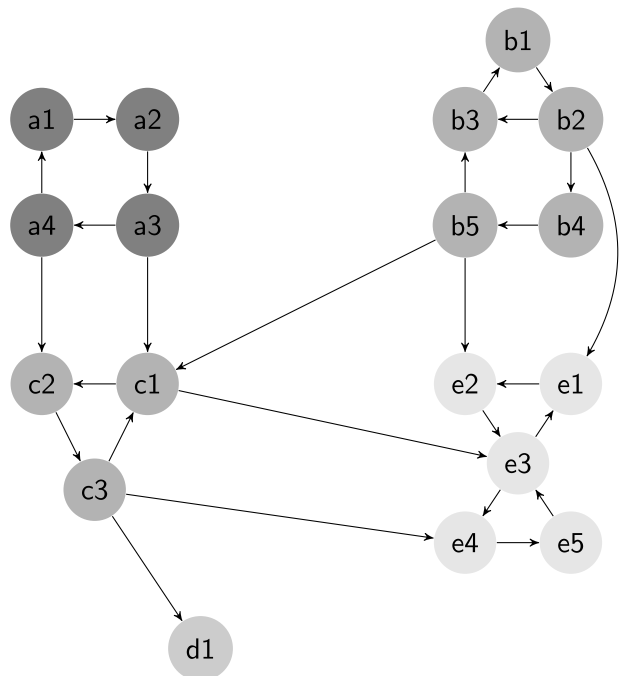

Figure 7. Example of a DAG

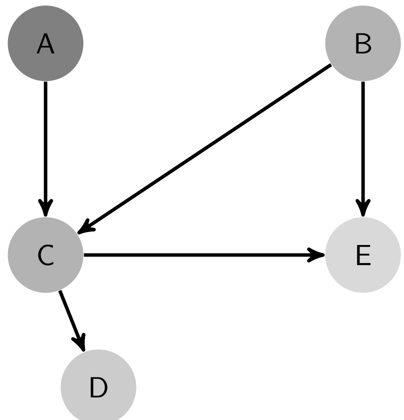

Figure 8. Condensation of the DAG

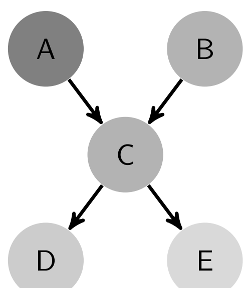

Figure 9. Transitive reduction of the DAG

The MSCCNE algorithm is a simple algorithm that exploits a network decomposition technique to find all the pure–strategy Nash equilibria of a financial network. It is worth noting that the MSCCNE algorithm can be easily adapted to compute the clearing payment vector of Eisenberg and Noe (2001).

Corollary 2. *Assume that the first-period endowment of each agent i is zero—that is, $z_i^1 = 0$. Then the MSCCNE algorithm computes the clearing payment vector in Eisenberg and Noe (2001).*

Recall that the clearing payment vector of Eisenberg and Noe (2001) is unique under mild conditions. Hence the existence of cyclical financial interconnections, while necessary for multiple equilibria, is not sufficient.

At the heart of the seminal contribution of Eisenberg and Noe (2001) lies the elegant *fictitious default algorithm* that computes the unique clearing payment vector. The fictitious default algorithm goes as follows. First, determine the set of agents who cannot fulfill their obligation, even when we assume that all agents receive their due payments. These agents will be called the *first wave of default.* Then, assume that the agents in the first wave of default pay their liabilities pro rata and the new defaulting agents will be called the *second wave of default* and so on until the algorithm terminates. In this way, the fictitious default algorithm produces a natural measure of systemic risk, which is the number of waves required to induce a given agent to default.

Echenique (2007) provides the most efficient algorithm for computing all pure–strategy Nash equilibria in the class of games of strategic complements, of which the default game is a special case. The algorithm elegantly checks whether there is another Nash equilibrium once the smallest and largest pure–strategy Nash equilibria are computed from classical algorithms (for example, Topkis (1979)).

While each of the above algorithms is clearly interesting in many aspects, arguably, the advantage of the MSCCNE algorithm developed in this paper is that it relies on the financial network architecture to compute the Nash equilibria. Generally, algorithms that exploit the financial network structure such as the algorithm developed in this paper, as well as having a clear computational advantage, provide valuable information on the strategic interactions among agents, as we will show below.

Corollary 3. *Let $\mathcal{W} = \{W_1, \dots, W_k\}$ denote the unique partition generated by the transitive reduction. Given two Nash equilibria $x = (x_{W_1}, \dots, x_{W_k})$ and $x' = (x'_{W_1}, \dots, x'_{W_k})$ of the default game. Then it holds that*

$$\inf\{x, x'\} = \{\min\{x_{W_1}, x'_{W_1}\}, \dots, \min\{x_{W_k}, x'_{W_k}\}\}$$

and

$$\sup\{x, x'\} = \{\max\{x_{W_1}, x'_{W_1}\}, \dots, \max\{x_{W_k}, x'_{W_k}\}\}$$

are also Nash equilibria of the default game.

Corollary 3 shows that the MSCCNE algorithm developed in this paper based on the concept of ear decomposition provides a further insight on the structure of the set of pure strategy Nash equilibria. More specifically, it holds that the infimum (resp. supremum) of two Nash equilibria is the product of the minimum of the corresponding Nash equilibria within each SCC. since Nash equilibria within each SCC are ranked.

5. Policy Implications of central clearing

From a policy perspective, in view of the multiplicity of Nash equilibria of the default game, there is the central policy question of equilibrium selection. In particular, it may be desirable to implement the best equilibrium in order to achieve financial stability and minimise the cost of default.

Given the best and the worst equilibria, agents in the network can be classified into three types:[8]

(1) agents that choose 0 in the worst equilibrium and 1 in the best equilibrium;
(2) agents that choose 0 in the worst equilibrium and 0 in the best equilibrium;
(3) agents that choose 1 in the worst equilibrium and 1 in the best equilibrium.

Note that agents of type (2) and (3) are not strategically relevant since they play the same action in the worst and the best equilibrium. Actually, we could construct a *reduced financial network* containing only agents of type (1). To do so, we first eliminate all outgoing links emanating from agents of type (3) and, since none of them defaults, add their liabilities pro rata to the cash flow of the agents intercepting their outgoing links. As for agents of type (2), given that they default and pay their inflows—i.e. their cash flow and the payments they receive from their debtors—they can be eliminated from the network by adding their cash flow to the cash flow of their creditors pro rata and by extending their ingoing liabilities links to their creditors pro rata so that the new liabilities directly link between their debtors and their creditors.

Recently, CCP has become increasingly the cornerstone of policy reform in financial markets. Introducing a CCP in the financial network modifies the structure of the financial

[8]Obviously, it is not possible for an agent to choose 1 in the worst equilibrium and 0 in the best.

network: each liability between a debtor and a creditor is erased and replaced by two new liabilities—one liability between the debtor and the CCP, and another one between the CCP and the creditor. As a consequence, one of the key benefits of central clearing is that, by breaking down the cyclical connections of financial liabilities, it reduces the aggregate level of default exposure, which in turn reduces default contagion.

There is a growing literature which investigates the benefits of central clearing. Duffie and Zhu (2011) show that CCP's reduce significantly the counterparty risk even when clearing across multiple derivative classes. Zawadowski (2013) suggests that a CCP eliminates *ex ante* own default externalities by making banks contribute to the insurance of counterparty risk in the form of a guarantee fund. In other respect, Tirole (2011) argues that centralisation should be encouraged and CCP's enhance transparency and allow for multilateral netting. Acharya and Bisin (2014) study how the lack of transparency between agents sharing default risk produce counterparty risk externality and show that this externality disappears when introducing a centralized clearing mechanism which ensures transparency. They prove that the main advantage of central clearing is enhancing the aggregation of information.

The following proposition points out another potential benefit of introducing central clearing in financial markets.

Proposition 5. *Introducing a CCP in each SCC of the reduced financial network achieves the best equilibrium in the default game at no additional cost.*

Proposition 5 shows that when a CCP intermediates the liabilities of each SCC of the reduced financial network,[9] the best equilibrium is achieved and the CCP is budget neutral. As a consequence, in addition to reducing default contagion by eliminating the cyclical financial interconnections, central clearing can also serve as a coordination device that achieves the best equilibrium of the default game.

The following example illustrates this point.

Example 2 Consider an economy of six agents connected through their ownership of each other's liabilities, among which only the first five agents are strategically relevant. Agents' endowments in the first period are $\mathbf{z}^1 = (22, 22, 75, 170, 100)$ and in the second

[9]That is, the financial network with only strategic relevant agents.

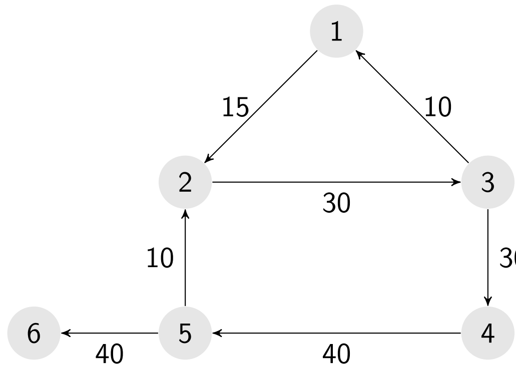

Figure 10. A financial network with five agents

period are $\mathbf{z}^2 = (3, 3, 3, 3, 3)$ and the interest rate is $r = 0.1$. All agents have the same utility function $U_i(e_i^1, e_i^2) = e_i^1 + e_i^2$. The financial liabilities of agents to each other are illustrated in the network in Figure 9.

This financial network contains a unique SCC $\{1, 2, 3, 4, 5\}$. To compute the Nash equilibria, we apply the USCCNE algorithm described above. We find three Nash equilibria—the best equilibrium $1, 1, 1, 1, 1$, the intermediate equilibrium $0, 0, 0, 1, 1$, and the worst equilibrium $0, 0, 0, 0, 0$—which we illustrate in Figures 10-12.

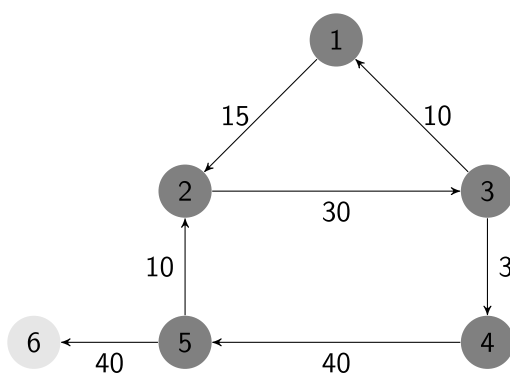

Figure 11. The best equilibrium

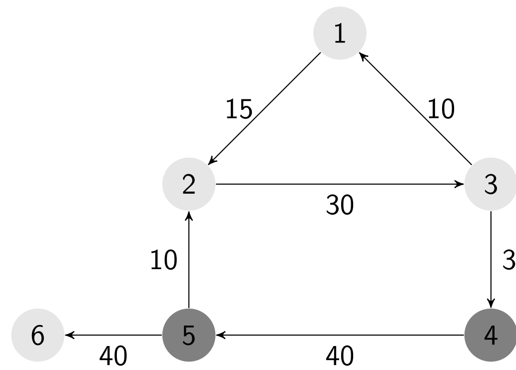

Figure 12. The intermediate equilibrium

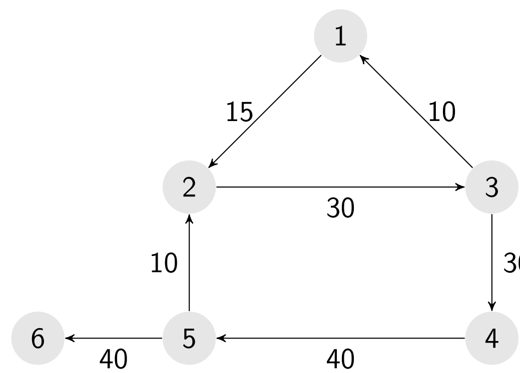

Figure 13. The worst Equilibrium

Adding a CCP will result in a new financial network as shown in Figure 13, with the following liabilities vector:

$$\tilde{\mathbf{L}} = (5, 5, 10, 10, 10, -40)\,.$$

Given that there are no feedback effects in the presence of the CCP, the minimum cash flow for an agent i to escape default is equal to the new liability $\tilde{L}_i$. Therefore, after the introduction of a CCP, it is easy to check that the best equilibrium is implemented at no additional cost since the inflows and outflows of CCP are equal.

6. Conclusion

This paper shows that the introduction of a CCP allows agents playing different actions at different Nash equilibria to achieve the best equilibrium at no additional cost. As a

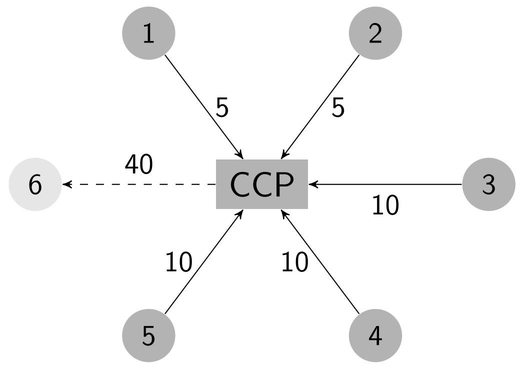

Figure 14. Adding a CCP

consequence, central clearing can serve as a coordination device in financial markets. While our result reinforces the key role CCP plays in financial markets, as highlighted in several important contributions by Duffie and Zhu (2011), Tirole (2011), Zawadowski (2013) and Acharya and Bisin (2014), it remains to be seen whether other policies can be designed to minimise the number of defaults, such as identifying key agents and targeting them through either cash injection or minimum endowment requirement.

7. Appendix

Proof of Proposition 1. Recall that the default game corresponds to a binary coordination game with two actions (Default) = 0 and (Non-Default) = 1 among which agents must choose.

First, for each agent i we will show that $\tau_i(\mathbf{a}_{-i})$ is well-defined given other agents' actions $\mathbf{a}_{-i} \in \{0,1\}^{N-1}$. To do so, for each agent i we consider an auxiliary economy with a modified network of liabilities, where we eliminate all outgoing links emanating from agent i and add his liabilities pro rata to the cash flow of the agents intercepting his outgoing links. Hence, the matrix of relative liabilities in the auxiliary economy is $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_{kj})_{k,j\in N}$, where $\hat{\alpha}_{kj} = \alpha_{kj}$ if $k \neq i$ and $\hat{\alpha}_{kj} = 0$ otherwise. Moreover, the (augmented) second-period endowment of agent j in the auxiliary economy is $\hat{z}_j^2 = z_j^2 + \alpha_{ij} L_i$.

Now, given other agents' actions $\mathbf{a}_{-i}$, let $\mathbf{x}^{\mathbf{a}_{-i}} = (x_j^{\mathbf{a}_{-i}})_{j\in N}$ denote the agents' storage strategies, where $x_j^{\mathbf{a}_{-i}} = z_j^1$ for each agent $j \neq i$ such that $a_j = 1$, and $x_j^{\mathbf{a}_{-i}} = 0$ otherwise. Let also $\boldsymbol{\pi}^{\mathbf{x}^{\mathbf{a}_{-i}}} = (\pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}})_{j\in N}$ denote the clearing payment vector, uniquely defined as in

Eisenberg and Noe (2001), such that for each agent j it holds that

$$\pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}} = \min\left\{\hat{z}_j^2 + (1+r)\, x_j^{\mathbf{a}_{-i}} + \sum_{k=1}^{n} \hat{\alpha}_{kj} \pi_k^{\mathbf{x}^{\mathbf{a}_{-i}}}; L_j\right\}.$$

Therefore, since $x_i^{\mathbf{a}_{-i}} = 0$ it holds that

$$\tau_i(\mathbf{a}_{-i}) = \max\left\{L_i - z_i^2 - \sum_{j=1}^{n} \hat{\alpha}_{ji} \pi_j^{\mathbf{x}^{\mathbf{a}_{-i}}}; 0\right\}. \tag{7.1}$$

Hence, the threshold $\tau_i(\mathbf{a}_{-i})$ is well-defined.

Moreover, it follows from Lemma 5 in Eisenberg and Noe (2001) (see, also, Theorem 6 in Milgrom and Roberts (1990)) that $\boldsymbol{\pi}^{\mathbf{x}^{\mathbf{a}_{-i}}}$ is increasing in $\mathbf{x}^{\mathbf{a}_{-i}}$, which, in turn, is increasing in $\mathbf{a}_{-i}$. Hence, it follows from (7.1) that the threshold $\tau_i(\mathbf{a}_{-i})$ is decreasing in $\mathbf{a}_{-i}$.□

Proof of Theorem 1. Since the threshold $\tau_i(\mathbf{a}_{-i})$ is decreasing in $\mathbf{a}_{-i}$ it follows that the best reply function of agent i

$$\Psi_i(\mathbf{a}_{-\mathbf{i}}) = \begin{cases} 1 & \text{if } r z_i^1 - \tau_i(\mathbf{a}_{-i}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is increasing in $\mathbf{a}_{-\mathbf{i}}$. By the Knaster–Tarski Theorem, there exists a fixed point of the following map:

$$\Psi : \{0,1\}^N \longrightarrow \{0,1\}^N$$

$$\Psi(\mathbf{a}) = (\Psi_1(\mathbf{a}_{-\mathbf{1}}), ..., \Psi_n(\mathbf{a}_{-\mathbf{n}})),$$

which will be a Nash equilibrium of the default game.□

Algorithm 1. (USCCNE) *Denote the i^{th} agent of ear j as v_{ji}. BFS refers to Breadth-First-Search.*

(1) Compute an ear decomposition of the network.
(2) For each ear E_j in $(E_p, E_{p-1}, ..., E_1)$
    (a) Begin a BFS from v_{j1}: For each visited agent u, if all of the agent's in-neighbours are in $\{V_{\text{visited}} \cup v_{j1}\}$, then

(i) Add u to T_j and to V_{visited}.

(ii) Calculate the activation outflow from v_{j1} that is sufficient for u not to default, conditional upon the previous activation levels of agents in T_j. If u is the parent node of an existing tree T_k, then calculate the activation outflows from v_{j1} that activate any interior strategy profiles in $\Gamma^*(T_k)$.

Add calculated activation outflows to list $A_j = \{(u, a_j(u)), ...\}$.

When all remaining out-neighbours of T_j have in-neighbours outside of $\{V_{\text{visited}} \cup v_{j1}\}$, terminate the BFS.

(b) Sort the agents and strategy profiles in T_{j1} by their activation outflows from v_{j1}.

(c) For each activation outflow in A_j, calculate tree-consistent strategy profiles for T_j, $\Gamma^*(T_{j1})$.

(d) For each strategy profile $\gamma \in \Gamma^*(T_j)$, calculate the activation inflow into v_{j1} that activates the strategy profile.

When all agents in the network are in V_{visited}, terminate the algorithm.

(3) For each strategy profiles in $\Gamma^*(T_j)$, calculate repayments and verify that the profile is an equilibrium. Drop any profiles that are not equilibria, leaving Γ^*.[10]

Proof of Proposition 3. Part (i): Let γ be an equilibrium strategy profile. If γ is not in $\Gamma^*(T_{v_{11}})$, then it must be the case that there is a subset of nodes V' such that $\gamma(V')$ is not tree-consistent for some tree $T_{v_{k1}}$. This would be a contradiction, as tree-inconsistency implies that there is no inflow into v_{k1} that would yield $\gamma(V')$ as a best response strategy profile.

Part (ii): Part (ii) is ensured by Step (2) of USCCNE. If step 1 generates any tree-consistent strategy profiles that are not equilibria of the default game, these will be identified upon the calculation of equilibrium repayments.□

[10]Repayments are calculated as follows:

$$\pi = (\mathbf{I} - \mathbf{D}\alpha')^{-1}((\mathbf{I} - \mathbf{D})\mathbf{L} + \mathbf{D}\mathbf{z}^2)$$

where $\mathbf{D}$ is a diagonal matrix with entry $D_{jj} = 1$ if and only if j is a defaulting agent. A strategy profile γ is removed if the calculated repayments are not consistent with the default decisions specified in γ.

Proof of Proposition 4. The breadth-first-searches conducted in (A.1.a) visit each edge in the network once. They may visit an individual node more than once, if that node has multiple in-neighbors, but each visit occurs via a different in-neighbor. There are a maximum of n^2 edges in a network, or alternatively, $|E| + n - 1$ where $|E|$ is the total number of ears. Steps (A.1.a.i-ii) are time complexity $\mathcal{O}(0)$, due to the recursive calculation of activation levels.

Step (A.1.b) requires a sorting of a list of a maximum of n agents, which incurrs worst-case time complexity n^2. Note that the sort will be faster when individual trees are smaller, as the time to sort an individual tree T_k and merge with sorted trees $T_{k+1}, T_{k+2}, ...$ is $\mathcal{O}(|T_k|^2) + \mathcal{O}(|T_{k+1}| + |T_{k+2}| + ...)$, where $|T_k|$ denotes the number of agents in tree T_k. Steps (A.1.c-d) are time complexity $\mathcal{O}(|T_k|)$.

The strategy profiles $\Gamma^*(T_1)$ generated in (A) form a total order over the space $\{0, 1\}^n$. It follows that $|\Gamma^*(T_1)| \leq n$. Step B can be performed in $\mathcal{O}(n^3)$ for each strategy profile, for example by solving a system of linear equations.□

Proof of Proposition 2. Suppose not—that is, the default game has multiple equilibria and the financial network does not have cyclical obligations. Let R denote the set of agents who play 0 in the worst Nash equilibrium and 1 in the best Nash equilibrium. Then the subnetwork induced by R contains an agent i that does not have any ingoing link. As a consequence, the inflow of agent i does not change between the worst equilibrium and the best equilibrium, and as a result agent i will not change his choice in the worst equilibrium and the best equilibium. This is a contradiction.□

Proof of Proposition 5. Adding a CCP in the middle of the financial network will net out the liabilities and will sort agents into two types: debtors and creditors to the CCP. Let node 0 represent the CCP, and $\tilde{L}_{i0}$ the liabilities to/from the CCP such that

$$\tilde{L}_{i0} = \sum_{j \in N} L_{ij} - \sum_{j \in N} L_{ji}.$$

Hence, if $\tilde{L}_{i0}$ is positive (resp. negative), agent i is a debtor (resp. creditor) to the CCP.

Since the best equilibrium can be reached, it follows that whenever agent i receives all the liabilities from his debtors, he will choose not default. Therefore, it holds that

$$z_i^2 + (1+r)\, z_i^1 + \sum_{j \in N} L_{ji} \geq \sum_{j \in N} L_{ij},$$

which implies

$$z_i^2 + (1+r)\, z_i^1 \geq \tilde{L}_{i0}.$$

Hence, the non-default condition is satisfied for each agent in the network with liabilities intermediated by the CCP and the best equilibrium is reached.□

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