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Abstract

In this paper, we develop a new game theoretic network centrality measure based on the Shapley value. To do so, we consider a coalitional game, where the worth of each coalition is the total play in the game introduced in Ballester et al. (2006). We first establish that the game is convex. As a consequence, the Shapley value belongs to the core, which enhances the attractive features of our new centrality measure. Then, we compute the Shapley value for various examples and illustrate some of its properties.

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1 Introduction

The large size and complexity of social networks has focused attention on searching methods to understand and extract valuable information from these network structures. Centrality analysis, is one of the important methods for analysing the structure of social networks. The objective of a centrality measure is to assign a ranking to the nodes that captures the structural properties as well as the underlying interactions. In the literature, a multitude of centrality measures have been proposed to provide insights into potential importance, power, and influence of nodes in social interactions. Perhaps, the most intuitive centrality measure is degree centrality, defined as the number of direct neighbours in the network, which gives importance to individuals with more connections.

In a key contribution Ballester et al. (2006) show that the Nash equilibrium efforts in a non-cooperative network game corresponds the Bonacich centrality. Unlike degree centrality, which overlooks indirect influences, Bonacich centrality is a global network measure since it accounts for influences from distant neighbours. Furthermore, Ballester et al. (2006) provide a characterisation of the key player, whose removal yields the highest reduction in the overall Nash equilibrium efforts, as the one with the highest intercentrality measure introduced in their paper. The intercentrality takes into account both direct and indirect effects of the removal of a player on the overall Nash equilibrium efforts.

In this paper, we pursue the novel approach of Ballester et al. (2006) to generate centrality measures from network games. More specifically, we show that solution concepts in cooperative game theory describing fair and stable outcomes of cooperation, can provide a rich framework for social network analysis. Chief among them, the Shapley value (Shapley, 1953), which assigns to each player his expected marginal contribution to all coalitions. It measures how important is each player to the overall cooperation, and what value can it reasonably expect. An equally important concept is the core (Shapley, 1955; Zhao, 2018), which represent feasible outcomes that cannot be improved upon by a group of players. While the Shapley value always exists and is unique, the core may consist of several outcomes or be empty, both Shapley value and core are used in numerous applications.

We consider a coalitional game, called the Bonacich Coalitional game, where the worth of each coalition is the overall Nash equilibrium efforts of the game introduced in Ballester et al. (2006) played on the network induced by the coalition. The uniqueness of the Nash equilibrium of the network game for each coalition implies that the BC game is well-defined. In addition, armed with our insight that the intercentrality measure corresponds to the marginal contribution of a player to a coalition, we show that the BC game is convex. For convex games the Shapley value is the barycenter of the core. The Shapley value of a game does not generally have to be in the core of the game, nor even be individually rational. However, there is a set-valued extension of the Shapley value, the Weber set, which always contains the core. Convex games are exactly those games for which the core and the Weber set coincide. Hence, for such games the Shapley value is an attractive core selection.

This paper contributes to game theoretic network centrality literature by developing a new network centrality measure based on the Shapley value of the associated BC game. Due to the correspondence between the intercentrality measure and the marginal contribution of a player to a coalition, and the convexity of the BC game, this measure turns out to be the average of all the intercentralities in the network with respect to all possible orders of the players. Consequently, it is called the Bonacich Shapley centrality.

The paper is organised as follows. In the next section we review the literature related to the Shapley value. In Section 3, we define the Bonacich Coalitional game and prove that the marginal contribution of a player increases as the coalition grows (e.g., it is convex). In Section 4 we present the Bonacich Shapley centrality measure and compute it for various examples and illustrate some of its properties. Section 5 concludes the paper.

2 Related literature

The Shapley value is a "Crown Jewel" of cooperative game theory (Thomson, 2019). It has many desirable properties and it is also the unique allocation rule satisfying some subset of these properties (see Moulin, 2014). For example, it is the unique allocation rule satisfying the properties of Equal treatment of equals (Symmetry), Additivity and Dummy player (Shapley, 1953). In this sense, we can say that it is a "fair" allocation.

One of the first economic aplications of the Shapley value can be found in the cost allocation literature. There are two classical cost allocation problems which applies the Shapley value as the best way to allocate the total cost. The first one is the Airport Problem (Littlechild and Owen, 1973). In that problem, an airport needs to built in order to accommodate a range of aircrafs which require differents lengths of runway. The Shapley value spreads the marginal cost of each required length of runway among all the airlines needing a runway of at least that long. In the end, airlines requiring a shorter runway pay less, and those needing a longer runway pay more. However, none of the airlines pay as much as they would have if they had chosen not to cooperate. The second example is the procedure to determine the rates of the telephone calls at Cornell University developed by Billera et al. (1978). The authors construct a game, where each calling instant is a player, that measures the minimal cost of servicing the demands given by a coalition. To solve the rates problem, they applied the Aumann and Shapley (1974) method to this game. The Shapley value has also been applied to other fields such as Management Accounting (Meca and Varela-Pena (2018) and Meca et al. (2019)), Telecommunications or Multi-agent Systems in Engineering (Sánchez-Soriano, 2019). Moretti and Patrone (2008) provides a survey about several applications of the Shapley value to very diverse fields showing the pertinence of the Shapley value to address real-life problems.

Closely related to this paper is the use of the Shapley value in networks. Lindelauf et al. (2013) use the Shapley value to identify key players in terrorist networks. They illustrate their game theoretic model through two case studies: Jemaah Islamiyah's

Bali bombing and Al Qaeda's 9/11 attack. Cesari et al. (2017) applies the Shapley value to quantify the potential of a gene in preserving the regulatory activity across all possible subsets of genes in a co-expression network. Subsequently, Cesari et al. (2018) show that the ability of the Shapley value to single out relevant genes in a co-expression network from the literature, is comparable to the one of other classical centrality measures. Gómez et al. (2003) introduces a new family of centrality measures for social networks by using coalitional games. The authors aim to measure centrality as variation in the power due to the social structure, using the Shapley value. Flores et al. (2019) proposes to use the Generalized Shapley value, introduced by Marichal et al. (2007), as a priori evaluation of the prospects of a group of players when acting as a group without imposing on the other players any concrete coalition structure. They considered two scenarios: (1) the analysis of criminal or terrorist organizations, where the police want to identify a small group of criminals or terrorists to neutralize in order to break up the criminal organization, (2) the analysis of formal and informal social networks in an organization, as well as the employee participation in virtual communities of practice for seeking knowledge.

As far as we know, none of the afore mentioned papers has considered the centrality of Bonacich for social networks by using coalitional games. The novelty of our model lies in analizing the noncooperative effort game from Ballester et al. (2006) as a cooperative game played on the network induced by the coalitions. Then, through this coalitional game, we construct a new centrality measure based on the Shapley value that is called the Bonacich Shapley centrality. It is the average of all the intercentralities in the network with respect to all possible orders of the players.

3 Bonacich Coalitional game

We consider a network $\mathbf{g}$ of $N = \{1, 2, ..., n\}$ players represented by an adjacency matrix $\mathbf{G}(N)$, where $g_{ij} = 1$ indicates a link between players i and j, and $g_{ij} = 0$ otherwise. Since the adjacency matrix $\mathbf{G}(N)$ is symmetric and non-negative it follows that its eigenvalues are real and the maximum eigenvalue $\lambda_{\max}(N)$ is positive and dominates in magnitude all other eigenvalues.

For any coalition of players $S \subseteq N$, let $\mathbf{g}(S)$ denote the subnetwork induced by S, with adjacency matrix $\mathbf{G}(S)$. Given a scalar $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)}\right[$ we define the matrix

$$\mathbf{M}(\mathbf{g}(S), \delta) = [\mathbf{I}_S - \delta \mathbf{G}(S)]^{-1} = \sum_{k=0}^{\infty} \delta^k \mathbf{G}^k(S). \tag{1}$$

The matrix $\mathbf{M}(\mathbf{g}(S), \delta)$ is well-defined since from the interlacing eigenvalue Theorem it holds that $\lambda_{\max}(N) = \max_{S \subset N} \lambda_{\max}(S)$. Note that each entry $m_{ij}(\mathbf{g}(S), \delta) = \sum_{k=0}^{\infty} \delta^k \mathbf{g}_{ij}^k(S)$ of $\mathbf{M}(\mathbf{g}(S), \delta)$ counts the number of walks in $\mathbf{g}(S)$ that start in i and end at j weighted by δ^k. In interpretation, the parameter δ is a damping/attenuation factor that scales down the relative weight of longer walks.

Define the vector of *Bonacich centrality* as

$$\mathbf{b}(\mathbf{g}(S), \delta) = [\mathbf{I}_S - \delta \mathbf{G}(S)]^{-1} \cdot \mathbf{1}_S, \tag{2}$$

where $\mathbf{1}_S$ is the $|S|$-dimensional vector of ones. The Bonacich centrality was put forth by Bonacich (1987) as a measure of prestige and importance. For player i, the Bonacich centrality is given by $b_i(\mathbf{g}(S), \delta) = \sum_{j \in S} m_{ij}(\mathbf{g}(S), \delta)$ and counts the total numbers of walks in $\mathbf{g}(S)$ that starts at i.[1]

We consider now a Bonacich Coalitional game, henceforth BC game, (N, δ, v_B), where N is the set of players, $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)}\right[$ is a decay/damping/attenuation factor, and $v_B : \mathscr{P}(N) \to \mathbb{R}$ is the so-called characteristic function of the game, which assigns to each coalition $S \subseteq N$ the sum of Bonacich centralities of the players in the coalition. That is,

$$v_B(S) := b(\mathbf{g}(S), \delta) = \sum_{i \in S} b_i(\mathbf{g}(S), \delta). \tag{3}$$

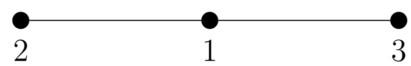
Note that for any coalition $S \subseteq N, v_B(S) > 0$.

[1] Note that, by definition, $m_{ii}(\mathbf{g}(S), \delta) \geq 1$ and thus $b_i(\mathbf{g}(S), \delta) \geq 1$, with $\mathbf{b}_i(\mathbf{g}(S), 0) = 1$.

Example 3.1. *Consider a network* $\mathbf{g}$ *with* $N = \{1, 2, 3\}$ *and* $\mathbf{G}(N)$ *given by*

$$\mathbf{G}(N) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We may notice that it is a star network with player 1 connected to all others. That is,



For any $0 \leq \delta < 0.7 = 1/\sqrt{2}$, *the BC game* (N, δ, v_B) *is given in the following table:*

S	$\mathbf{M}(\mathbf{g}(\mathbf{S}), \delta)$	$v_B(S)$
$\{i\}$	1	1
$\{2,3\}$	$\mathbf{I}_2$	2
$\{1,2\}$ $\{1,3\}$	$\begin{pmatrix} \frac{1}{1-\delta^2} & \frac{\delta}{1-\delta^2} \\ \frac{\delta}{1-\delta^2} & \frac{1}{1-\delta^2} \end{pmatrix}$	$\frac{2}{1-\delta}$
$\{1,2,3\}$	$\begin{pmatrix} \frac{1}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} \\ \frac{\delta}{1-2\delta^2} & \frac{1-\delta^2}{1-2\delta^2} & \frac{\delta^2}{1-2\delta^2} \\ \frac{\delta}{1-2\delta^2} & \frac{\delta^2}{1-2\delta^2} & \frac{1-\delta^2}{1-2\delta^2} \end{pmatrix}$	$\frac{3+4\delta}{1-2\delta^2}$

An allocation rule for BC games is a vector $\psi(v_B) \in \mathbb{R}^n_+$ to such that,

$$\sum_{i \in N} \psi_i(v_B) = v_B(N).$$

Each component ψ_i indicates the value allocated to $i \in N$, so an allocation rule for BC games is a procedure to allocate the maximum Bonacich centrality among the players.

An important set of allocations for BC games is the *core*. It is formed by all those allocations that are coalitionally stable. That is, allocations with the property that the sum of the values allocated to the members of each coalition S is at least $v(S)$. For a BC game (N, δ, v_B), the core will be denoted by $C(v_B)$, and its elements will be called *core allocations.*

We say the BC game is *convex* if for all $S, T \subseteq N$ such that $S \subseteq T$ with $i \in S$,

$$v_B(S) - v_B(S \backslash \{i\}) \leq v_B(T) - v_B(T \backslash \{i\}).$$

The convexity property provides additional information about the game: the marginal contribution of a player increases as the coalition grows. It is well-known as the snowball effect.

Ballester et al. (2006) show that the marginal contribution of a player i to coalition $S \subseteq N$ is equal the intercentrality of player in the network $\mathbf{g}$ induced by S. That is,

$$v_B(S) - v_B(S \backslash \{i\}) = c_i(\mathbf{g}(S), \delta) := \frac{b_i(\mathbf{g}(S), \delta)^2}{m_{ii}(\mathbf{g}(S), \delta)}. \tag{4}$$

This makes the marginal contribution of a player to a coalition a readily interpreted network measure. Note that the intercentrality, unlike Bonacich centrality, accounts also for the loops through player i. For instance, holding $b_i(\mathbf{g}(S), \delta)$ fixed (as in regular networks) $c_i(\mathbf{g}(S), \delta)$ decreases with player i's loops $m_{ii}(\mathbf{g}(S), \delta)$.

The main theorem in this paper states that BC games are always convex. Thus BC games are monotonic.[2]

Theorem 3.2. *Every BC game is convex.*

Proof. The proof of Theorem 3.2, together with all of our other proofs, appears in the Appendix.□

It is a well-known result in cooperative game theory that the core of a convex game is non-empty. That means that we can always find coalitional stable and consistent allocation rules for BC games. In addition, convex games are the only

[2]It is well-known that every non negative convex game is always monotonic

ones whose core coincides with the Weber set; that is, it is the convex hull of the marginal contributions vectors for the different orders/permutations of the players in N. The core of convex games also coincides with the bargaining set of Maschler et al. (1971). Based on the relationship between the intercentrality and the marginal contributions of its corresponding BC game (see (4)), we show here that the extreme points of the core of a BC game are vectors of intercentralities.

We denote by $\Pi(N)$ the set of all possible orders in N. Take $\sigma = (i_1, i_2, i_3, ..., i_n) \in \Pi(N)$. The vector of marginal contributions with respect to the order σ, is $m^\sigma(v_B) \in \mathbb{R}^n$, with

$$m^\sigma_{i_1}(v_B) := v_B(\{i_1\}),$$
$$m^\sigma_{i_2}(v_B) := v_B(\{i_1, i_2\}) - v_B(\{i_1\}),$$
$$m^\sigma_{i_3}(v_B) := v_B(\{i_1, i_2, i_3\}) - v_B(\{i_1, i_2\}),$$
....
$$m^\sigma_{i_{n-1}}(v_B) := v_B(\{i_1, i_2, i_3, ..., i_{n-1}\}) - v_B(\{i_1, i_2, i_3, ..., i_{n-2}\}),$$
$$m^\sigma_{i_n}(v_B) := v_B(\{i_1, i_2, i_3, ..., i_n\}) - v_B(\{i_1, i_2, i_3, ..., i_{n-1}\}).$$

Notice that there are at most $n!$ marginal contribution vectors $m^\sigma(v_B)$. The Weber set, $W(v_B)$, is the convex hull of the marginal contributions vectors for the different orders of the players in N, that is

$$W(v_B) = conv\{m^\sigma(v_B)/\sigma \in \Pi(N)\}.$$

Next proposition states that the core of BC games is the convex hull of intercentralities vectors for all orders in N.

Given an order $\sigma \in \Pi(N)$, we denote by $P^\sigma_i \subseteq N$ the set of the predecessors of player i, including itself, in σ. Notice that $P^\sigma_i \neq \emptyset$ and $P^\sigma_i = \{i\}$, if player i is first in order σ. The vector of inter-centralities in network is $c^\sigma(\mathbf{g}(N), \delta) \in \mathbb{R}^n$, with $c^\sigma_i(\mathbf{g}(N), \delta) := c_i(\mathbf{g}(P^\sigma_i), \delta)$.

Proposition 3.3. *The core of every BC game is the convex hull of vectors of inte-centralities in the network, for the different orders of the players in N. That is,*

$$C(v_B) = conv\{c^{\sigma}(\mathbf{g}(N), \delta) \mid \sigma \in \Pi(N)\}. \tag{5}$$

A very natural allocation rule for BC games is the vector of Bonacich centralities in the grand coalition, $\mathbf{b}(\mathbf{g}(N), \delta)$. It is a core allocation. Indeed, for every $S \subseteq N$, by (9) it holds that,

$$\sum_{i \in S} b_i(\mathbf{g}(N), \delta) \geq \sum_{i \in S} b_i(\mathbf{g}(S), \delta) = v_B(S).$$

Hence, $\mathbf{b}(\mathbf{g}(N), \delta) \in C(v_B)$. But the core of BC games is large enough to find other core allocations, for instance, in terms of marginal contributions/inter-centralities, as shown in the next Section.

4 Bonacich Shapley centrality

Since BC games are convex, cooperative game theory provides well know allocation rules for them with good properties, as coalitional stability (core allocations) and acceptability. We highlight the Shapley value, which assigns a unique allocation, among the players, of a total surplus generated by the grand coalition. As we already mentioned, the Shapley value measures how important is each player to the overall cooperation, and what value can it reasonably expect.

Formally, the Shapley value, $\phi(v_B) \in \mathbb{R}^n_+$ is given by

$$\phi_i(v_B) := \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \left(v_B(S) - v_B(S \backslash \{i\})\right). \tag{6}$$

Next example compares both allocation rules, the vector of Bonacich centralities in the grand coalition and the Shapley value, for a star network with three players.

Example 4.1. *Consider again the star network with $N = \{1, 2, 3\}$ and $\mathbf{G}(N)$ given in example 3.1. It follows from convexity that the core of (N, δ, v_B) is the convex* *hull of the following extreme points:*

$$(1, \frac{1+\delta}{1-\delta}, \frac{1+\delta-4\delta^2}{(1-\delta)(1-2\delta^2)}), (1, \frac{1+\delta-4\delta^2}{(1-\delta)(1-2\delta^2)}, \frac{1+\delta}{1-\delta}), (\frac{1+4\delta+4\delta^2}{1-2\delta^2}, 1, \frac{1+\delta-4\delta^2}{(1-\delta)(1-2\delta^2)})$$

$$(\frac{1+\delta}{1-\delta}, 1, 1), (\frac{1+\delta}{1-\delta}, \frac{1+\delta-4\delta^2}{(1-\delta)(1-2\delta^2)}), 1), (\frac{1+4\delta+4\delta^2}{1-2\delta^2}, 1, 1).$$

The vector of Bonacich centralities in the grand coalition is

$$\mathbf{b}(\mathbf{g}(N), \delta) = (\frac{1+2\delta}{1-2\delta^2}, \frac{1+\delta}{1-2\delta^2}, \frac{1+\delta}{1-2\delta^2}).$$

We obtain from (6) the Shapley value for each player. Note that players 2 *and* 3 *are symmetric. Then,*

$$\phi(v_B) = \frac{1}{6}\left(\frac{6+6\delta-8\delta^2-4\delta^3}{(1-\delta)(1-2\delta^2)}, \frac{6-9\delta^2+4\delta^3}{(1-\delta)(1-2\delta^2)}, \frac{6-9\delta^2+4\delta^3}{(1-\delta)(1-2\delta^2)}\right),$$

Note that neither $\mathbf{b}(\mathbf{g}(N), \delta)$ *nor* $\phi(v_B)$ *are extreme points for the core. Actually, the Shapley value is the average of the above extreme points for the core, i.e. it is the barycenter of the core.*

It is well-known that the Shapley value of any cooperative game can be obtained as the average of all vector of marginal contributions with respect to any possible order of the players. It is the barycenter of the core. That is,

$$\phi(v_B) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v_B). \tag{7}$$

Next proposition shows that the Shapley value of a player in a BC game is the average of all the inter-centralities in the network with respect to all possible orders of the players.

Proposition 4.2. *The Shapley value for each player* $i \in N$ *is given by*

$$\phi_i(v_B) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} c_i^\sigma(\mathbf{g}(N), \delta). \tag{8}$$

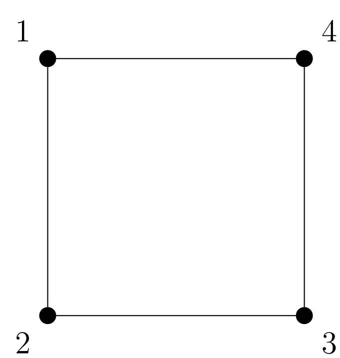
Proof. It is obtained directly by applying Proposition 3.3, that is, $m^\sigma(v_B) = c^\sigma(\mathbf{g}(N), \delta)$, for all $\sigma \in \Pi(N)$, in (7). □

The next two examples show that for some networks the Shapley value, for the corresponding BC game, coincides with the Bonacich centrality in the grand coalition and they are equal to the egalitarian allocation.

Example 4.3. *Consider a regular network* $\mathbf{g}$ *with* $N = \{1, 2, 3, 4\}$ *and* $\mathbf{G}(N)$ *given by*

$$\mathbf{G}(N) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that the players who are directly connected on this network are {1,2}, {1,4}, {2,3} and {3,4}. That is,



The BC game (N, δ, v_B), *for any* $\delta \in [0, 1/2[$, *is given in the following table:*

S	$\mathbf{M}(\mathbf{g}(\mathbf{S}),\delta)$	$v_B(S)$
$\{i\}$	1	1
$\{1,3\}$ $\{2,4\}$	$\mathbf{I}_2$	2
$\{1,2\}$ $\{1,4\}$ $\{2,3\}$ $\{3,4\}$	$\begin{pmatrix} \frac{1}{1-\delta^2} & \frac{\delta}{1-\delta^2} \\ \frac{\delta}{1-\delta^2} & \frac{1}{1-\delta^2} \end{pmatrix}$	$\frac{2}{1-\delta}$
$\{1,2,4\}$ $\{1,3,4\}$	$\begin{pmatrix} \frac{1}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} \\ \frac{\delta}{1-2\delta^2} & \frac{1-\delta^2}{1-2\delta^2} & \frac{\delta^2}{1-2\delta^2} \\ \frac{\delta}{1-2\delta^2} & \frac{\delta^2}{1-2\delta^2} & \frac{1-\delta^2}{1-2\delta^2} \end{pmatrix}$	$\frac{3+4\delta}{1-2\delta^2}$
$\{1,2,3\}$ $\{2,3,4\}$	$\begin{pmatrix} \frac{1-\delta^2}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} & \frac{\delta^2}{1-2\delta^2} \\ \frac{\delta}{1-2\delta^2} & \frac{1}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} \\ \frac{\delta^2}{1-2\delta^2} & \frac{\delta}{1-2\delta^2} & \frac{1-\delta^2}{1-2\delta^2} \end{pmatrix}$	$\frac{3+4\delta}{1-2\delta^2}$
N	$\begin{pmatrix} \frac{1-2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} & \frac{2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} \\ \frac{\delta}{1-4\delta^2} & \frac{1-2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} & \frac{2\delta^2}{1-4\delta^2} \\ \frac{2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} & \frac{1-2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} \\ \frac{\delta}{1-4\delta^2} & \frac{2\delta^2}{1-4\delta^2} & \frac{\delta}{1-4\delta^2} & \frac{1-2\delta^2}{1-4\delta^2} \end{pmatrix}$	$\frac{4}{1-2\delta}$

It's easy to check that the Bonacich centrality measure matches the egalitarian allocation. That is, for all $i \in N$,

$$b_i(\mathbf{g}(N),\delta) = \frac{1}{1-2\delta}.$$

On the other hand, we can prove (just playing with the orders $\sigma \in \Pi(\{1,2,3,4\})$ that the Shapley value for any player $i \in N$ is

$$\phi_i(v_B) = \frac{1}{4}\cdot c_i(\mathbf{g}(\{i\}),\delta) + \frac{1}{4}\cdot c_i(\mathbf{g}(N),\delta) + \frac{1}{12}\cdot \left[\sum_{j \in N\setminus\{i\}} c_i(\mathbf{g}(\{i,j\}),\delta) + \sum_{j,k \in N\setminus\{i\}} c_i(\mathbf{g}(\{i,j,k\}),\delta) \right].$$

Considering $c_i(\mathbf{g}(\{i\}), \delta) = 1$, for all $i \in N$, and generalizing the last term, we obtain

$$\phi_i(v_B) = \frac{1}{4} + \frac{1}{4} \cdot c_i(\mathbf{g}(N), \delta) + \frac{1}{12} \sum_{S \subset N \setminus \{i\}; 1 \leq |S| \leq 2} c_i(\mathbf{g}(S \cup \{i\}), \delta).$$

Doing some algebra, we obtain that for all $i \in N$,

$$\phi_i(\mathbf{g}(N), \delta) = \frac{1}{1 - 2\delta}.$$

Thus, the Shapley value for the BC game associated to this regular network matches the Bonacich centrality and coincides with the egalitarian allocation[3]*.*

Example 4.4. *Consider a network $\mathbf{g}$ with $N = \{1, 2, ..., n\}$ and $\mathbf{G}(N)$ given by $\mathbf{g}_{ij}(N) = 1$, for all $i, j \in N$. It is a complete network with n players.*

We know that,

$$\mathbf{I} - \delta\mathbf{G} = \begin{pmatrix} 1 & -\delta & \dots & -\delta \\ -\delta & 1 & \dots & -\delta \\ \vdots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ -\delta & -\delta & \dots & 1 \end{pmatrix} = (1+\delta)(\mathbf{I} - \frac{\delta}{1+\delta}\mathbf{1}^T\mathbf{1}).$$

From the Sherman-Morrison formula (see Sherman and Morrison, 1950), it follows that

$$(\mathbf{I} - \delta\mathbf{G})^{-1} = ((1+\delta)(\mathbf{I} - \frac{\delta}{1+\delta}\mathbf{1}^T\mathbf{1}))^{-1} = \frac{1}{(1+\delta)(1-(n-1)\delta)}((1-(n-2)\delta)\mathbf{I} + \delta\mathbf{G}).$$

Hence,

$$m_{ii}(\mathbf{g}(N), \delta) = \frac{1-(n-2)\delta}{(1+\delta)(1-(n-1)\delta)},$$

$$b_i(\mathbf{g}(N), \delta) = \frac{1}{1-(n-1)\delta},$$

[3] Note that player 1 and 3 are symmetric as well as player 2 and 4. However, neither players 1 and 2 nor players 3 and 4 are symmetric. However, the Shapley value here is equal for all of them.

and

$$c_i(\mathbf{g}(N),\delta) = \frac{b_i(\mathbf{g}(N),\delta)^2}{m_{ii}(\mathbf{g}(N),\delta)} = \frac{1+\delta}{(1-(n-1)\delta)(1-(n-2)\delta)}$$

It is easy to check that for any coalition $S \subseteq N$, and any player $i \in S$,

$$b_i(\mathbf{g}(S),\delta) = \frac{1}{1-(|S|-1)\delta},$$

and

$$c_i(\mathbf{g}(S),\delta) = \frac{1+\delta}{(1-(|S|-1)\delta)(1-(|S|-2)\delta)}.$$

The associated BC game is given by

$$v_B(S) = \frac{|S|}{1-(|S|-1)\delta}$$

for any $S \subseteq N$ and $0 \leq \delta < min_{1<|S|\leq n}\left(\frac{1}{|S|-1}\right) = \frac{1}{n-1}$. It is a symetric game. The Shapley value matches the vector of Bonacich centralities for the grand coalition and they are equal to the egalitarian allocation, e.g. for all $i \in N$

$$\phi_i(v_B) = b_i(\mathbf{g}(N),\delta) = \frac{1}{1-(n-1)\delta}.$$

In this paper we propose a new game theoretic centrality measure, *the Bonacich Shapley centrality.* It is the Shapley value of BC games.

Definition 4.5. *Let $\mathbf{g}$ be a network of $N = \{1,2,...,n\}$ players and (N,δ,v_B) the associated BC game with $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)}\right[$. For any player $i \in N$, the Bonacich Shapley centrality is given by*

$$BS_i(\mathbf{g},\delta) =: \phi_i(v_B) = \frac{1}{n} + \frac{1}{n}c_i(\mathbf{g}(N),\delta) + \sum_{i\in S\subset N;|S|=s\geq 2} \gamma(s)\cdot c_i(\mathbf{g}(S),\delta),$$

with $\gamma(s) = \frac{(s-1)!(n-s)!}{n!}$.

The last example compares the Bonacich centrality and the Bonacich Shapley centrality in the example given by Ballester et al. (2006).

Example 4.6. *Consider the network* $\mathbf{g}$ *in Figure 1 with* $n = 11$ *players.*

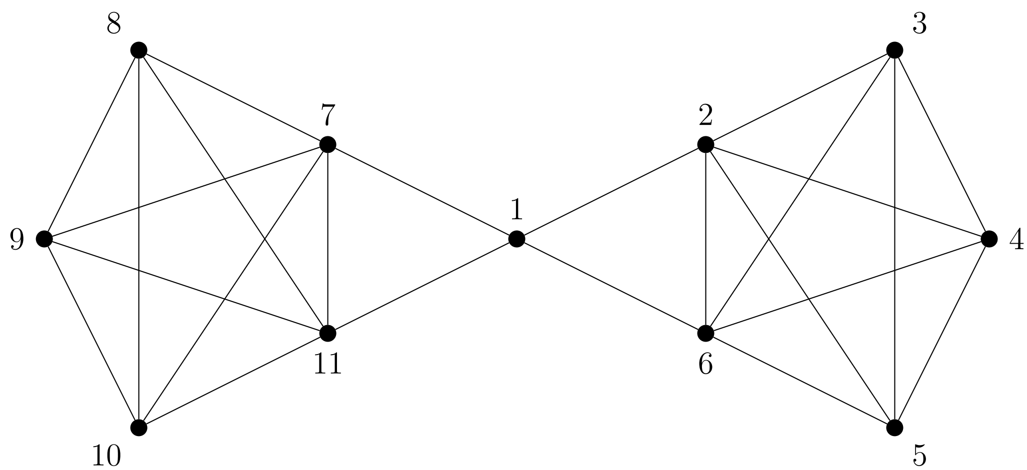


Figure 1. 11-players network

The next two tables list the Bonacich centrality, the interecentrality, and the Bonacich Shapley centrality for two values of δ*. Note that* $\lambda_{\max}(N) = 4.7$*, and so* $\delta \in [0, 0.212[$ *. There are three type of players. The only player of type* 1 *is node* 1*. Players of type* 2 *correspond to nodes* 2, 6, 7, *and* 11*; and type 3 players correspond to nodes* 3, 4, 5, 8, 9, *and* 10*. An asterisk identifies the highest column value.*

Player type	$b_i(\mathbf{g}(N), 0.1)$	$BS_i(\mathbf{g}, 0.1)$	$c_i(\mathbf{g}(N), 0.1)$
1	1.75	1.74	2.92
2	1.88*	1.90*	3.28*
3	1.72	1.71	2.79

Table 1. Centrality measures for $\delta = 0.1$

Player type	$b_i(\mathbf{g}(N), 0.2)$	$BS_i(\mathbf{g}, 0.2)$	$c_i(\mathbf{g}(N), 0.2)$
1	8.33	8.87	41.67*
2	9.17*	9.25*	40.33
3	7.78	7.63	32.67

Table 2. Centrality measures for $\delta = 0.2$

Player 2 has the highest Bonacich centrality for both low and high δ. However, when δ is high, player 1 has the highest intercentrality due to direct and indirect effects in the grand coalition. Our computation shows that player 2 has also the highest Bonacich Shapley centrality for both low and high δ, which is in line with Bonacich centrality. In interpretation, player 2 has the highest average direct and indirect effects on aggregate outcome of all possible coalitions.

5 Conclusions

Our paper provides a new method to construct network centrality measure based on Shapley value to analyze peer-effects and team production in networks a la Ballester et al (2006). The Shapley value enjoys several attractive properties and various axiomatisations. Our analysis shows that the properties of Shapley value could be further complemented as its key ingredient of marginal contribution is readily interpreted from network analysis. It remains to be seen whether our setting could accommodate more general externalities than network team production (see, for example Hellmann (2020)). Equally, it remains to be seen whether other key solution concepts in game theory with different emphasis than the Shapley value (e.g., the Nucleoulus or a proportional rule) could be amenable (or meaningful) to network analysis.

6 Appendix

Proof of Theorem 3.2 Let's consider $i \in S \subseteq T \subseteq N$. Recall that

$$\mathbf{b}(\mathbf{g}(S), \delta) = [\mathbf{I_S} - \delta\mathbf{G}(\mathbf{S})]^{-1}\mathbf{1_S} = \begin{bmatrix} 1 & -\delta\ \mathbf{G}_{i,S\backslash\{i\}} \\ -\delta\ \mathbf{G}^T_{i,S\backslash\{i\}} & \mathbf{I}_{S\backslash\{i\}} - \delta\mathbf{G}(S\backslash\{i\}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \mathbf{1}_{S\backslash\{i\}} \end{bmatrix}$$

Using block matrix inversion formula:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}$$

we obtain that

$$m_{ii}(\mathbf{g}(S),\delta) = (1-\delta\mathbf{G}^T_{i,S\backslash\{i\}}(\mathbf{I}_{S\backslash\{i\}}-\delta\mathbf{G}(S\backslash\{i\}))^{-1}\delta\mathbf{G}_{i,S\backslash\{i\}})^{-1}$$

$$= \frac{1}{1-\delta^2\sum_{j,k\in S\backslash\{i\}} m_{jk}(\mathbf{g}(S\backslash\{i\}),\delta)}$$

and

$$\mathbf{b}_i(\mathbf{g}(S),\delta) = m_{ii}(\mathbf{g}(S),\delta)(1+\delta\mathbf{G}^T_{i,S\backslash\{i\}}(\mathbf{I}_{S\backslash\{i\}}-\delta\mathbf{G}(S\backslash\{i\}))^{-1}\mathbf{1}_{S\backslash\{i\}})$$

$$= m_{ii}(\mathbf{g}(S),\delta)(1+\delta\sum_{j\in S\backslash\{i\}}\mathbf{b}_j(\mathbf{g}(S\backslash\{i\}),\delta)).$$

$$= m_{ii}(\mathbf{g}(S),\delta)(1+\delta\sum_{j,k\in S\backslash\{i\}} m_{jk}(\mathbf{g}(S\backslash\{i\}),\delta)).$$

Therefore,

$$\frac{\mathbf{b}_i(\mathbf{g}(S),\delta)^2}{m_{ii}(\mathbf{g}(S),\delta)} = m_{ii}(\mathbf{g}(S),\delta)(1+\delta\sum_{j,k\in S\backslash\{i\}} m_{jk}(\mathbf{g}(S\backslash\{i\}),\delta))^2$$

Hence, it is enough to prove that for all $i, j \in S \subseteq T \subseteq N$,

$$m_{ij}(\mathbf{g}(S),\delta) \leq m_{ij}(\mathbf{g}(T),\delta) \tag{9}$$

to obtain

$$\frac{\mathbf{b}_i(\mathbf{g}(S),\delta)^2}{m_{ii}(\mathbf{g}(S),\delta)} \leq \frac{\mathbf{b}_i(\mathbf{g}(T),\delta)^2}{m_{ii}(\mathbf{g}(T),\delta)},$$

which proves convexity. Indeed, recall that

$$m_{ij}(\mathbf{g}(S),\delta)=\sum_{k=0}^{\infty}\delta^k\mathbf{g}_{ij}^k(S),$$

where $\mathbf{g}_{ij}^k(S)$ denotes the number of walks in S starting from i terminating at j of length k. For $i,j\in S\subseteq T\subseteq N$, let $\mathbf{g}_{ij}^k(S^{out})$ denote the number of walks in T starting from i terminating at j of length k and not fully contained in S. Observe that

$$\mathbf{g}_{ij}^k(T)=\mathbf{g}_{ij}^k(S)+\mathbf{g}_{ij}^k(S^{out}),$$

which implies

$$\mathbf{g}_{ij}^k(S)\leq\mathbf{g}_{ij}^k(T),$$

and so (9) holds.□

Proof of Proposition 3.3 Consider a BC game (N,δ,v_B), with $\delta\in\left[0,\frac{1}{\lambda_{\max}(N)}\right[$. We know by Theorem 3.2 that (N,δ,v_B) is convex. Then, the core coincides with the Weber set (see Shapley, 1971), that is $C(v_B)=conv\{m^\sigma(v_B)/\sigma\in\Pi(N)\}$.

Take an order $\sigma=(i_1,i_2,i_3,...,i_n)\in\Pi(N)$. Then, we know that

$$\begin{aligned}
&m_{i_1}^\sigma(v_B)=c_{i_1}(\mathbf{g}(\{i_1\}),\delta)=1,\\
&m_{i_2}^\sigma(v_B)=c_{i_2}(\mathbf{g}(\{i_1,i_2\}),\delta),\\
&m_{i_3}^\sigma(v_B)=c_{i_3}(\mathbf{g}(\{i_1,i_2,i_3\}),\delta),\\
&\ldots\ldots\ldots\ldots\ldots\\
&m_{i_{n-1}}^\sigma(v_B)=c_{i_{n-1}}(\mathbf{g}(\{i_1,i_2,i_3,...,i_{n-1}\}),\delta),\\
&m_{i_n}^\sigma(v_B)=c_{i_n}(\mathbf{g}(\{i_1,i_2,i_3,...,i_n\}),\delta).
\end{aligned}$$

Hence, for all player $i\in N$, $c_i^\sigma(\mathbf{g}(N),\delta)=c_i(\mathbf{g}(P_i^\sigma),\delta)=m_i^\sigma(v_B)$. We can conclude then that, for all $\sigma\in\Pi(N), m^\sigma(v_B)=c^\sigma(\mathbf{g}(N),\delta)$.□

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References

[1] Aumann R.J. and Shapley, L.S. (1974) Values of non-atomic games. Princeton University Press, Princeton.

[2] Ballester C., Calvó-Armengol A. and Zenou, Y. (2006) Who's who in networks. Wanted: the key player. Econometrica 74: 1403-1417.

[3] Billera L.J. and Heath D.C. (1982) Allocation of Shared Costs: A Set of Axioms Yielding a Unique Procedure. Mathematics of Operations Research 7: 32-39.

[4] Bonacich P. (1987) Power and centrality: a family of measures. American Journal of Sociology 92: 1170-1182.

[5] Cesari G., Algaba E., Moretti S. and Nepomuceno J.A. (2017) A game theoretic neighbourhood-based relevance index. In: Cherifi C, Cherifi H, Karsai M, Musolesi M (eds). Complex Networks and Their Applications VI. Studies in Computational Intelligence. Springer, Cham Vol. 689.

[6] Cesari G., Algaba E., Moretti S. and Nepomuceno J.A. (2018). An application of the Shapley value to the analysis of co-expression networks. Applied Network Science 3:35.

[7] Flores R., Molina E. and Tejada J. (2019). The Shapley Value as a Tool for Evaluating Groups: Axiomatization and Applications. Capter 13 in in E. Algaba,

F.V. Fragnelli and J. Sánchez-Soriano (Eds.), Handbook of the Shapley value. London, United Kindon: CRC Press. Taylor and Francis Group.

[8] Gómez D., González-Aranguena E., Manuel C., Owen G., del Pozo M. and Tejada, J. (2003). Centrality and power in social networks: a game theoretic approach. Mathematical Social Sciences 46: 27-54.

[9] González-Díaz J., García-Jurado I. and Fiestras-Janeiro M.G. (2010) An introductory course on mathematical game theory. Graduate Studies in Mathematics, vol 115. American Mathematical Society.

[10] Hellmann T. (2021) Pairwise stable networks in homogeneous societies with weak link externalities. European Journal of Operational Research 291 (3): 1164-1179.

[11] Lindelauf R.H.A., Hamers H.J.M., Husslage B.G.M. (2013) Cooperative game theoretic centrality analysis of terrorist networks: The cases of Jemaah Islamiyah and Al Qaeda. European Journal of Operational Research 229: 230-238.

[12] Littlechild S. C. and Owen G. (1973). A Simple Expression for the Shapley Value in a Special Case. Management Science. 20 (3): 370-372.

[13] Marichal J.L., Kojadinovic I. and Fujimoto, J. (2007) Axiomatic characterizations of generalized values. Discrete Applied Mathematics 155: 26-43.

[14] Maschler M., Peleg B. and Shapley L.S. (1971) The kernel and bargaining set for convex games. International Journal of Game Theory 1: 73-93.

[15] Meca A. and Varela-Pena J.C. (2018) Corporation Tax Games: An Application of Linear Cost Games to Managerial Cost Accounting. In: Game Theory in Management Accounting, pp. 361-378. Springer.

[16] Meca A., García-Martínez J.A. and Mayor-Serra A.J. (2019). The Shapley value of corporation tax games with dual benefactors. Chapter 15 in E. Algaba, F.V. Fragnelli and J. Sánchez-Soriano (Eds.), Handbook of the Shapley value. London, United Kindon: CRC Press. Taylor and Francis Group.

[17] Moretti S. and Patrone, F. (2008) Transversality of the Shapley value. TOP 16,:1-41.

[18] Moulin H. (2014) Cooperative microeconomics: a game-theoretic introduction. Princeton University Press, Princeton, New Jersey.

[19] Sanchez-Soriano, J. (2019) The Shapley value in Telecommunication Problems. Chapter 16 in E. Algaba, F.V. Fragnelli and J. Sánchez-Soriano (Eds.), Handbook of the Shapley value. London, United Kindon: CRC Press. Taylor and Francis Group.

[20] Shapley L.S. (1953) A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (eds) Contributions to the Theory of Games II. Annals of Mathematics Studies, vol 28. Princeton University Press, Princeton, pp. 307- 317.

[21] Shapley L.S. (1955) Markets as Cooperative Games Rand Corporation Paper P-629: 1-5.

[22] Shapley L.S. (1971) Cores of convex games. International journal of game theory 1(1): 11-26.

[23] Sherman J. and Morrison W.J. (1950). Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix. Annals of Mathematical Statistics 21 (1): 124-127.

[24] Thomson W. (2019). The Shapley Value, a Crown Jewel of Cooperative Game Theory. Chapter 1 in E. Algaba, F.V. Fragnelli and J. Sánchez-Soriano (Eds.), Handbook of the Shapley value. London, United Kindon: CRC Press. Taylor and Francis Group.

[25] Zhao J. (2018) Three little-known and yet still significant contributions of Lloyd Shapley. Games and Economic Behaviour 108: 592-599.