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Constrained public goods in networks

Nizar Allouch and Maia King

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Nizar Allouch[†]
University of Kent

Maia King[‡]
University of Oxford

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Abstract

This paper analyses the private provision of public goods where agents interact within a fixed network structure and may benefit only from their direct neighbours' provisions. We survey the literature and then generalise the public goods in networks model of Bramoullé and Kranton (2007) to allow for constrained provision. In so doing, we show that, using the concept of k-insulated set, any network supports a Nash equilibrium with no intermediate contributors.

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[†]School of Economics. E-mail: `n.allouch@kent.ac.uk`

[‡]Blavatnik School of Government. E-mail: `maia.king@bsg.ox.ac.uk`

Non-Technical Summary

Voluntary contributions account for the provision of many public goods, ranging from essential infrastructure, education, to health care, while at the aggregate level charitable giving represents a significant proportion of GDP in many countries. The seminal contribution of Bergstrom, Blume, and Varian (1986), built on an earlier striking result by Warr (1981), provides a rigorous investigation of the standard model of private provision of pure public goods.

Recent work on public goods in networks, initiated by the key paper of Bramoullé and Kranton (2007), has many interesting facets and applications. The technology of network analysis allows us to generalise from the provision of pure public goods, which benefit all agents, to a more detailed model of local public goods with a heterogeneous benefit structure shaped by a network.

This paper analyses the private provision of public goods where agents interact within a fixed network structure and may benefit only from their direct neighbours' provisions. We survey the literature and then generalise the public goods in networks model of Bramoullé and Kranton (2007) to allow for constrained provision. In so doing, we show that, any network supports a Nash equilibrium with no intermediate contributors.

1 Introduction

Voluntary contributions account for the provision of many public goods, ranging from essential infrastructure, education, to health care. The seminal contribution of Bergstrom, Blume, and Varian (1986, known as BBV) provides the groundwork for the analysis of the private provision of pure public goods. Although there are influential antecedents of Malinvaud (1972) and Becker (1974), BBV provide the most rigorous treatment of the private provision model and present some of its startling results, such as showing that as the economy grows large, the average contribution goes to zero and only the wealthy contribute. BBV also develop the well-known 'neutrality' results of Warr (1983), where exogenous income redistribution among contributors—that leaves the set of contributors unchanged—has no effect either on the aggregate provision of public goods or the consumption of private goods. The neutrality result, further analysed by Bernheim (1986) and Andreoni (1989), is related to complete crowding-out of budget neutral government provision, 'dollar-for-dollar', which provides sharp testable implications.

Recent work on public goods in networks, initiated by the key paper of Bramoullé and Kranton (2007), has many interesting facets and applications. The technology of network analysis allows us to generalise from the provision of pure public goods, which benefit all agents, to a more detailed model of local public goods with a heterogeneous benefit structure shaped by a network. In this paper, we generalise Bramoullé and Kranton's (2007) model to the case where each agent's public good contribution is constrained exogenously. Then we study Nash equilibria with no intermediate contributors—that is, Nash equilibria where agents are either full contributors or free-riders. Our analysis shows that these 'no-intermediate-contributors Nash equilibria' correspond to the k-insulated sets of the network.

As a special case, we obtain the key result of Bramoullé and Kranton (2007) that specialised Nash equilibria—that is, equilibria with both full contributors and free-riders—correspond to maximal independent sets of the network, where no node in a set is connected to each other. If fact the concept of k-insulated set is a generalisation of the concept of maximal independent set since maximal independent sets correspond to the particular case of $k = 0$. It is worth noting that while both specialised equilibria and no-intermediate-contributors Nash equilibria obviously rule out intermediate contributors, there is a key difference which is, unlike specialised equilibria, no-intermediate-contributors Nash equilibria may have free-riders. This key difference can be significant, since whenever the network is connected, no-intermediate-contributors Nash equilibria without free-riders are always stable, whereas stable specialised equilibria always have free-riders.

1.1 Survey

Unlike Tiebout's seminal contribution, and the subsequent vast literature on the local public good model, the public goods in networks model allows for geographic spillovers among nearby communities.[1] The public goods in networks literature has burgeoned to include more general approaches: Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) incorporate private information; Galeotti and Goyal (2010) investigate issues of network formation; Bramoullé, Kranton, and D'Amours (2014) investigate private provision with linear best replies; Allouch (2015) extends BBV analysis to networks; and Elliott and Golub (2015) explore decentralized mechanisms for efficient provision. Other important related contributions have been made by Acemoglu, García-Jimeno, and Robinson (2015), Acemoglu, Malekian, and Ozdaglar (2016), López-Pintado (2017), Kinateder and Merlino (2017), and Sun (2017).[2]

While the existence of a Nash equilibrium in the private provision model is guaranteed by Brouwer's fixed point theorem, BBV rely on the assumption of normality of private and public goods to establish the uniqueness of a Nash equilibrium. For public goods in networks, to prove uniqueness of a Nash equilibrium, Allouch (2015) introduces the network normality assumption which stipulates a standard normality of the private good and a strong normality of the public good, depending on the lowest eigenvalue of the adjacency matrix of the network. The proof is a generalisation of Bergstrom, Blume, and Varian (1986) for pure public goods and of Bramoullé, Kranton, and D'Amours (2014) game of linear strategic substitutes.[3]

Given the neutrality of government redistribution by lump-sum transfers, which is similar in motivation to the Second Welfare Theorem, a number of alternative private

[1] An exception is Bloch and Zenginobuz (2006).

[2] See, also, Ramachandran and Chaintreau (2015) and Bodwin (2017).

[3] For more recent contributions on the existence and uniqueness of a Nash equilibrium in network games, see Naghizadeh and Liu (2017), Melo (2018), and Parise and Ozdaglar (2018).

incentive mechanisms have been suggested to increase social welfare; for example, by Buchholz, Cornes, and Rübbelke (2011). Yet, as pointed out by Allouch (2015), neutrality often breaks down in general networks beyond a homogeneous structure of contributors, and hence government redistribution by lump-sum transfers can be restored as a mechanism to raise social welfare.

Furthermore, Allouch (2015, 2017) shows that the impact of income redistribution on aggregate consumption and welfare is determined by Bonacich centrality. Bonacich centrality, due to Bonacich (1987) is key concept which usually measures prestige, importance, and influence in social networks. It was first introduced to economics in a key paper by Ballester, Calvó-Armengol, and Zenou (2006), as being proportional to Nash equilibrium actions in a linear best-reply game, and has also been shown to be important in several other applications in the field. For public goods in networks, Bonacich centrality summarises information on each agent's impact on aggregate provision and welfare after income redistribution. Even though one of the most pervasive ideas in the theory of public goods is that they are always under-provided by a system of private provision, quite surprisingly, Allouch (2017) shows that a welfare-increasing income transfer—which goes from a high Bonacich centrality agent to a low Bonacich centrality agent—actually decreases aggregate provision. As a consequence, one may conclude that when public goods in networks are provided solely by voluntary contributions, raising social welfare and raising aggregate provision are sharply conflicting policy objectives. Meanwhile Allouch and King (2018) establish conditions for Pareto-improving transfers and show that income redistribution may surprisingly lead to a transfer paradox. Finally, Bourlès, Bramoullé, and Perez-Richet (2017) provide an important investigation of the neutrality of transfers in an altruistic setting.

2 Public goods in networks: constrained provision

Consider a model of public goods in networks—that is, local public goods with benefits accessible along geographic or social links. There are n agents arranged in a connected fixed network $\mathbf{g}$. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that agent i and agent j are neighbours in the network $\mathbf{g}$ and $g_{ij} = 0$

otherwise. In particular, we assume that $g_{ii} = 0$ for each agent i. We denote by $N = \{1, \ldots, n\}$ the set of agents and by $\mathcal{N}_i = \{j \in N \mid g_{ij} = 1\}$ the set of agent i's neighbours.

Each agent i faces a marginal cost $c > 0$ for providing a public good and his payoffs, for the profile of provisions $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$, are given by:

$$U_i(\mathbf{x}) = b(x_i + \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j) - cx_i,$$

where b is the benefit function, which is differentiable, strictly increasing, and concave such that $b'(k) = c,$ for any positive integer $k = 1, 2, 3, \ldots$.

We also assume that each agent i public good contribution is bounded $x_i \in [0, 1]$.

Proposition 1. *A profile of provisions $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$ is a Nash equilibrium if and only if for every agent i one of the following holds*

(1) $x_i = 0$ *and* $\sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j \geq k$.

(2) $x_i = 1$ *and* $\sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j \leq k - 1$.

(3) $x_i = k - \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j$ *and* $k - 1 < \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j < k$.

Proof. The best reply of each agent is

$$x_i = f_i(\mathbf{x}_{-i}) = \max\{\min\{k - \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j, 1\}, 0\}.$$

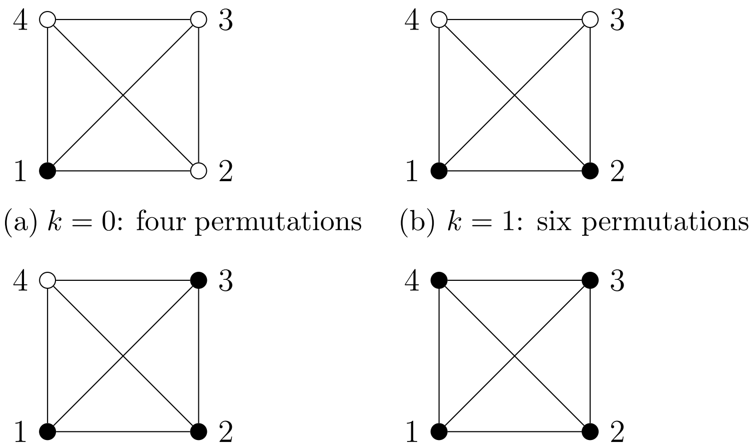
which gives the required results above. ■

Proposition 1 tells us the public good contribution of each agent at a Nash equilibrium. We may distinguish three types of agents, *free-riders*, who contribute 0, *full-contributors*, who contribute 1, and the others, *intermediate contributors*—that is, $0 < x_i < 1$. Using a standard fixed-point argument, it is easy to show that there exists a Nash equilibrium for any network.

2.1 No-intermediate-contributors Nash equilibrium

Further work is needed in order to relate the shape of contribution profiles to the underlying network. To do so, in the following, we will focus on Nash equilibria with no

Figure 1: k-insulated sets in the complete network of four agents



(a) $k = 0$: four permutations (b) $k = 1$: six permutations

(c) $k = 2$: four permutations (d) $k \geq 3$: one permutation

intermediate contributors, which we will call a *NIC* (*no-intermediate-contributors*) Nash equilibrium. We will use the following notion from graph theory:

Definition 1 (Jagota, Narasimhan, and Šoltés (2001)). *For a positive integer k, a k-insulated set of a network* $\mathbf{g}$ *is a set of players $S \subseteq N$ such that each agent in S is adjacent to at most k other players in S and each agent not in S is adjacent to at least $k+1$ players in S.*

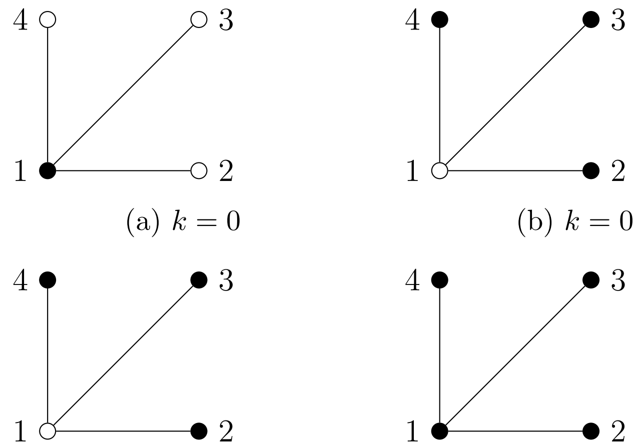
The concept of k-insulated set generalises maximal independent set since 0-insulated sets are exactly the maximal independent sets.[4]

Proposition 2. *A profile of contributions is a* NIC *Nash equilibrium if and only if its set of full contributors is a $(k-1)$-insulated set of the network* $\mathbf{g}$. *Since for every network* $\mathbf{g}$ *there exists a a $(k-1)$-insulated set, there always exists a NIC Nash equilibrium. Assign* 1 *to players in a $(k-1)$-insulated set and* 0 *to those outside.*

Proof. The proof follows easily from Proposition 1. ■

[4]Unlike maximal independent set, the existence of k-insulated sets is less obvious, and was established in Jagota, Narasimhan, and Šoltés (2001)

Figure 2: k-insulated sets in the star network of four agents



(a) $k = 0$ (b) $k = 0$ (c) $k = 1, 2$ (d) $k \geq 3$

Proposition 2 shows that network structure can always preclude a Nash equilibrium with intermediate contributions—that is, in every network structure there is a Nash equilibrium where no agent provides an intermediate contribution. Note also that all agents of degree at most $k-1$ must be in all $(k-1)$-insulated sets and as a consequence are full contributors at any *NIC* Nash equilibrium. In particular if $(k-1)$ is greater than the maximum degree of the network all agents are full contributors.

We will illustrate *NIC* Nash equilibrium in few canonical network structures: the complete network and the star network (see Figure 1 and Figure 2).

For a complete network, observe that the $(k-1)$-insulated sets correspond to subsets of agents of size k if $1 \leq k < N$ and to the entire set of players if $N \leq k$. Therefore, the number of *NIC* Nash equilibria increases initially and then decreases as k increases.[5]

For a star network, observe that there are two 0-insulated sets consisting of either the core and the periphery agents. If $k \geq 1$ then there exists only one k-insulated set (all periphery players for $k+1 < N$ and all players if $k+1 \geq N$). Therefore, the number of Nash equilibria decreases as k increases.

Since there could be many Nash equilibria (either *NIC* Nash equilibria or other Nash equilibria) in our setting, a simple notion of stability based on Nash tâtonnement, similar

[5]It is worth noting that while the number of contributors increases as k increases the number of *NIC* Nash equilibria does not always necessarily decrease.

to Bramoullé and Kranton (2007), can be used to reduce the number of equilibria in a natural way.[6]

Proposition 3. *Only* NIC *Nash equilibria without free-riders or where every free-rider is connected to at least* $(k + 1)$ *full contributors are stable.*

Similar to Bramoullé and Kranton (2007), the proof rests on the strategic substitutability of efforts of connected agents. First, consider an equilibrium where an agent i is an intermediate contributor. This means that at least one of his neighbours is also an intermediate contributor. Decrease the contribution of agent i by a small amount. His neighbour(s) will adjust by increasing their own efforts. This increase can lead i to reduce his effort even more, which will lead the system away from the initial equilibrium. Now, consider a *NIC* Nash equilibrium where there are no free-riders, or every free-rider is linked to at least $(k + 1)$ full contributors. Then it can be easily checked that the process described above no longer works since each full contributor agent will revert to his initial action.

Finally, recall that *NIC* Nash equilibria with $k = 0$ coincide with specialised equilibria in Bramoullé and Kranton (2007). More generally, it is worth noting that both specialised equilibria and *NIC* Nash equilibria obviously rule out intermediate contributors. Nonetheless, there is a key difference which is, unlike specialised equilibria, a *NIC* Nash equilibrium could be without free-riders. This key difference can be significant since whenever the network is connected, *NIC* Nash equilibria without free-riders are always stable, whereas stable specialised equilibria always have free-riders.

References

[1] Acemoglu, D., C. García-Jimeno, and J.A. Robinson (2015) "State capacity and economic development: a network approach," *American Economic Review* 105, 2364-2409.

[2] Acemoglu, D., A. Malekian, and A. Ozdaglar (2016) "Network security and contagion," *Journal of Economic Theory* 166, 536-585.

[6] See also Bervoets and Faure (2016), for a recent investigation of stability.

[3] Parise, F. and A. Ozdaglar (2018) "A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis," Working paper

[4] Allouch, N. (2015) "On the private provision of public goods on networks," *Journal of Economic Theory* 135, 527-552.

[5] Allouch, N. (2017) "The cost of segregation in (social) networks," *Games and Economic Behavior* 106,329-342.

[6] Allouch, N. and M. King (2018) "Policy reform in networks," Working paper

[7] Andreoni, J. (1989) "Giving with impure altruism: applications to charity and Ricardian equivalence," *Journal of Political Economy* 97, 1447-1458.

[8] Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006) "Who's who in networks. Wanted: The key player," *Econometrica* 74, 1403-1417.

[9] Becker, G. (1974) "A theory of social interactions," *Journal of Political Economy* 82, 1063-1093.

[10] Bergstrom, T., L. Blume, and H. Varian (1986) "On the private provision of public goods," *Journal of Public Economics* 29, 25-49.

[11] Bernheim, B.D. (1986) "On the voluntary and involuntary provision of public goods," *American Economic Review* 76, 789-793.

[12] Bervoets, S. and M, Faure (2016) "Best response in pure public good games in networks," Preprint.

[13] Bloch, F. and Ü. Zenginobuz (2006) "Tiebout equilibria in local public good economies with spillovers," *Journal of Public Economics* 97, 1447-1458.

[14] Bodwin, G. (2017) "Testing core membership in public goods economies," Working paper

[15] Bonacich, P. (1987) "Power and centrality: a family of measures," *American Journal of Sociology* 92, 1170-1182.

[16] Bourlès, R., Y. Bramoullé, and E. Perez-Richet (2017) "Altruism in networks," *Econometrica* 85, 675-689.

[17] Bramoullé, Y. and R. Kranton (2007) "Public goods in networks," *Journal of Economic Theory* 135, 478-494.

[18] Bramoullé, Y., R. Kranton, and M. D'Amours (2014) "Strategic interaction and networks," *American Economic Review* 104, 898-930.

[19] Buchholz, W., R. Cornes, and D. Rübbelke (2011) "Interior matching equilibria in a public good economy: an aggregative game approach," *Journal of Public Economics* 95, 639-645.

[20] Elliott, M. and B. Golub (2018) "A network approach to public goods," *Journal of Political Economy* Forthcoming,

[21] Galeotti, A. and S. Goyal (2010) "The law of the few," *American Economic Review* 100, 1468-1492.

[22] Galeotti, A., S. Goyal, M. Jackson, F. Vega-Redondo, and L. Yariv (2010) "Network games," *Review of Economic Studies* 77, 218-244.

[23] Jagota, N., G., Narasimhan, and L., Šoltés, (2001) "A Generalization of maximal independent sets" *Discrete Applied Mathematics,* 109, 223-235.

[24] Kinateder M. and L.P. Merlino (2017) "Public goods in endogenous networks," *American Economic Journal: Microeconomics* 9, 187-212.

[25] López-Pintado, D. (2017) "Influence networks and public goods," *SERIEs* 8, 97-112.

[26] Malinvaud, E. (1972) *Lectures on Microeconomic Theory*, North-Holland. 107

[27] Melo, E. (2018) *A Variational Approach to Network Games*, FEEM Working Paper

[28] Naghizadeh, P. and M. Liu (2017) "On the uniqueness and stability of equilibria of network games,"*Proceedings of the 55th Allerton conference*

[29] Sun, G-Z. (2017) *The Samuelson condition and the Lindahl scheme in networks,* Journal of Public Economics, 156, 73-80.

[30] Ramachandran, A. and A. Chaintreau (2015) "Who Contributes to the Knowledge Sharing Economy?" *In Proceedings of the 2015 ACM Conference on Online Social Networks*

[31] Warr, P.G. (1983) "The private provision of a public good is independent of the distribution of income," *Economics Letters* 13, 207-211.

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