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# Aggregation in Networks

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# AGGREGATION IN NETWORKS

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ABSTRACT. In this paper, we show that a concept of aggregation can hold in network games. Breaking up large networks into smaller pieces, which can be replaced by representative players, leads to a coarse-grained description of strategic interactions. This method of summarizing complex strategic interactions by simple ones can be applied to compute Nash equilibria. We also provide an application to public goods in networks to show the usefulness of our results. In particular, we highlight network architectures that cannot prevent free-riding in public good network games. Finally, we show that aggregation enhances the stability of a Nash equilibrium.

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## NON-TECHNICAL SUMMARY

Understanding, and making sense of, large economic networks is an increasingly important problem from an economic perspective, due to the ever-widening gap between technological advances in constructing such networks, and our ability to predict and estimate their properties. Throughout history, various concepts have been developed to reduce the inherent complexity found in large economic systems, thereby rendering them more amenable to economic analysis. One prominent example is aggregation, which aims to devise representative concepts that can be analyzed in a more tractable manner. For instance, a key question, which appeared in the seminal contributions of von Neumann and Morgenstern (1944), Chapter IX, Gorman (1953, 1961), and Shapley (1964, 1967), is: when does a group of individuals behave as if it were a single individual?

Our investigation of aggregation in network games is quite similar in motivation. Often, the reason such an argument holds in the above literature appears to hinge on having identical preferences or compositions. Our approach suggests that aggregation holds for a similar reason in network games; however, the homogeneity is brought about by the network architecture rather than behavior or structure.

Our findings could potentially have empirical applications to many network models in economics, including public goods and targeting/finding the key players' policies. Nonetheless, it remains to be seen whether other approaches from the vast and important literature on network position similarity, across myriad disciplines, ranging from biology and sociology to computer science - see, for example, Gagneur et al. (2004) and Newman (2006) - could be useful to further analyze complex strategic interactions.

## 1. INTRODUCTION

The economics of networks, which focuses on modeling and understanding varied economic interactions, has recently become one of the most active and dynamic fields in economics. It has the potential for important and lasting policy implications—see, for example, Goyal (2007) and Jackson (2008)—but it is notable that most economic interactions take place in large networks, whose sheer sizes and complex structures make economic analysis quite a challenging task. Nevertheless, it is well known that economic networks have a rich degree of symmetry due to similar linkage patterns for individuals having similar economic characteristics such as income, education, and preferences, for firms facing the same competitors, and for countries having similar bilateral agreement policies.

In this paper, we show that advantage can be taken of the symmetric features of economic networks. More specifically, we show that a concept of aggregation that ensures a group of players behaves like a single player holds for network games, a subject of ongoing research as in Ballester, Calvó-Armengol, and Zenou (2006) for externalities, Bramoullé and Kranton (2007) for public good provision, and Bramoullé, Kranton, and D’Amours (2014) for various economic interactions.<sup>1</sup> A key ingredient of our analysis is a group of players, called a module, such that players in the group have exactly the same neighbors outside the group.<sup>2</sup> In interpretation, since players in a module are indistinguishable by players outside the module in terms of their network position, outside players are affected either by their aggregate action or by nothing and hence one can substitute players in the group with a single representative player.

A concrete example, which arises naturally, is a set of firms competing domestically while facing the same overseas competition, which can then be replaced by just one big firm. Another concrete example is several countries privately providing a public good, such as cybersecurity, which could be made accessible via bilateral agreements. Assume that there is a group of countries with identical outside bilateral agreements. Therefore the group of countries can be represented by a single country regardless of the group’s architecture of bilateral agreements.

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<sup>1</sup>For recent related contributions, see also Acemoglu, Malekian, and Ozdaglar (2016), Elliott and Golub (2016), Günther and Hellmann (2017), and Kinateder and Merlino (2017).

<sup>2</sup>The notion of a modular set has been rediscovered several times in many fields including cooperative game theory by Shapley (1967) under the name of committee.

We first establish that a partition of players of a network game into modules gives rise to two-level *nested games*—a *module game* played within each module and a *composite game* played between representative players of the modules—such that each Nash equilibrium of the network game corresponds to a combination of Nash equilibria of the nested games. More specifically, the composite game determines the participation rates of the various modules, whose actions are the Nash equilibria of the module games. Thereby, we establish a systematic relationship between each player’s network position in a module, and his Nash equilibrium actions in the network game.

Furthermore, we show that a network game can be decomposed into a unique hierarchy of nested games. Key to this are the modules that overlap with no other modules, called strong modules, which, when ordered by inclusion, define a unique tree, called the modular decomposition tree, whose root is the set of players and whose leaves are the single players.<sup>3</sup> By fitting nested games into each other along the nodes of the modular decomposition tree, we obtain a unique hierarchical decomposition of the network game, which is useful for the analysis of strategic interactions. First, it determines the nature of strategic interactions between the representative players, ranging from strategic complements to strategic substitutes. Second, it can be used to carry out a recursive computation of Nash equilibria, which could be of great algorithmic interest.

We provide an application of our results to the model of public goods in networks, introduced in Bramoullé and Kranton (2007). The key question addressed in Bramoullé and Kranton (2007) is how the network architecture of spillovers influences public goods provision, in the absence of coordination. Our aggregation approach complements the analysis of Bramoullé and Kranton (2007), as it provides a necessary condition in order to have a Nash equilibrium with strictly positive contributions—that is, with no free-riders. Despite the attractive normative feature of sharing the burden of public goods among all players, such an equilibrium is not always guaranteed to exist. The necessary condition rules out the simultaneous presence of a single player and a non-single player in fully connected strong modules since this presence brings about a mismatch between what these players contribute and consume of the public goods, leading one of them to become a free-rider. This necessary condition for the existence of an equilibrium without free-riders, which also becomes sufficient for a special class of networks, illustrates the role played by the strong modules of the network in determining public goods provision.

<sup>3</sup>The concept of the modular decomposition tree was introduced in Gallai (1967). A similar decomposition also appeared in Shapley (1967).

In the last part of the paper, we provide another application of our results to games of strategic substitutes. We first relax the uniformity of link requirement of the modules so that outside players are affected by the same proportion of aggregate action or by nothing. Then, we investigate the stability of a Nash equilibrium. Interestingly, we show that aggregation increases the range of stability of a Nash equilibrium. A key finding of Bramoullé, Kranton, and D'Amours (2014) is that the stability range of symmetric Nash equilibria, which have received much attention in the literature, is relatively small. Therefore, aggregation can be especially useful for enlarging the range of stability of symmetric equilibria.

The paper is organized as follows. In Section 2, we present the basic model of network games. In Section 3, we introduce the concept of aggregation and nested games. In Section 4, we show that aggregation yields a unique hierarchy of nested games. In Section 5, we provide an application of our results to public goods in networks. Section 6 relates the stability of the various Nash equilibria. Section 7 concludes the paper.

## 2. THE MODEL

We consider a strategic form game  $\Gamma(\mathbf{g}, \delta)$  with  $N = \{1, \dots, n\}$  players embedded on an undirected and unweighted network  $\mathbf{g}$  of interactions, and where  $\delta \in [0, 1]$  measures how much player  $i$ 's action is affected by his neighbors' actions. Each player  $i$  chooses an action  $x_i \in \mathbb{R}_+$ . Given a subset of players  $I$  and a profile of actions  $\mathbf{x} = (x_1, \dots, x_n)$ , let  $\mathbf{x}_I = (x_i)_{i \in I}$  denote the actions of the players in  $I$  and  $x_I = \sum_{i \in I} x_i$  denote their sum. As usual, let  $\mathbf{x}_{-i} = \mathbf{x}_{N \setminus \{i\}}$  denote the actions of all other players than  $i$ . The payoffs of player  $i$  for the profile of actions  $\mathbf{x} = (x_1, \dots, x_n)$  are  $U_i(\mathbf{x}) = U_i(x_i, \mathbf{x}_{-i})$ . Player  $i$  seeks to maximize his payoffs and has a best-reply function

$$x_i = f_i(\mathbf{x}_{-i}) \stackrel{\text{def}}{=} \max\{1 - \delta x_{\mathcal{N}_i(\mathbf{g})}, 0\},$$

where  $\mathcal{N}_i(\mathbf{g})$  denotes  $i$ 's neighbors in  $\mathbf{g}$  and 1 is the action player  $i$  chooses in isolation.

As shown in Bramoullé, Kranton, and D'Amours (2014), this type of game,  $\Gamma(\mathbf{g}, \delta)$ , can be used to represent various types of economic interactions, including the model of public goods in networks, introduced in Bramoullé and Kranton (2007), and the model of negative externalities with linear-quadratic payoffs, introduced in Ballester, Calvo-Armengol, and Zenou (2006).

At a Nash equilibrium  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  of the game  $\Gamma(\mathbf{g}, \delta)$ , each player's action is a best-reply to his neighbors' actions, that is,  $x_i^* = f_i(\mathbf{x}_{-i}^*)$  for each player  $i \in N$ . The existence of a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  is guaranteed by Brouwer's fixed point theorem by restricting strategies of players to  $[0, 1]^n$ . As usual, let  $A$  denote the set of active players at the Nash equilibrium.

### 3. MODULAR AGGREGATION

We now introduce a network position similarity of a group of players, which ensures that it can behave like a single player. A group of players  $M$  is called a module if they have exactly the same neighbors outside the module, that is, for any player  $i \in N \setminus M$ , either  $i$  is adjacent to every player in  $M$  or  $i$  is adjacent to no player in  $M$ . It is easy to notice that each single player  $\{1\}, \dots, \{n\}$  and the entire set of players  $N = \{1, \dots, n\}$  are always modules, called trivial modules, which may well be the only modules for some networks. Connected components are also always modules.

A partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  of the set of players  $N$  is called a modular partition if  $M_k$  is a module of  $\mathbf{g}$ , for each  $k = 1, \dots, K$ .<sup>4</sup> Given two disjoint modules  $M_k$  and  $M_h$  of  $\mathbf{p}$ , either every player in  $M_k$  is a neighbor of every player in  $M_h$  or no player in  $M_k$  is adjacent to a player in  $M_h$ . Thus, the relationship between two disjoint modules is either adjacent or nonadjacent. Hence the modular partition  $\mathbf{p}$  gives rise to a new network,  $\mathbf{g}/\mathbf{p}$ , called the quotient network, whose vertices are the modules of the partition  $\mathbf{p}$  and links are the adjacencies of these modules.

Now we define a composite game played on the quotient network  $\mathbf{g}/\mathbf{p}$ , denoted by  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; \mathbf{z})$ , where  $\mathbf{z} = (z_1, \dots, z_K) \in \mathbb{R}_+^K$  is a vector of weights determined exogenously. This set-up means that in the quotient network, players positions are filled by representative players of the modules. For each module  $M_k$ , there is a representative player  $k$ , who chooses an action  $r_k \in [0, 1]$ . Representative player  $k$ 's payoffs depend on his own action  $r_k$  and the actions of the other representative players  $\mathbf{r}_{-k}$ . We denote the payoffs of the representative player  $k$  by  $V_k$ , which are assumed to yield the best-reply function:

$$r_k = F_k(\mathbf{r}_{-k}) \stackrel{\text{def}}{=} \max\{1 - \delta \sum_{h \in \mathcal{N}_k(\mathbf{g}/\mathbf{p})} z_h r_h, 0\}.$$

The following result shows that a Nash equilibrium of the network game corresponds to a combination of Nash equilibria of the (smaller) nested games—that is, a module game

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<sup>4</sup>Note that the modular partition may not be unique.

played within each module and a composite game played between the representative players of the modules.

**Theorem 1.** *Given a modular partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  of the set of players  $N$ , the following are equivalent:*

- (1)  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$
- (2)  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that
  - (a)  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, K$ , and
  - (b)  $\mathbf{r}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; y_{M_1}^*, \dots, y_{M_K}^*)$ .

Hence, it follows from Theorem 1 that finding the Nash equilibria of the nested games could provide significant insights into the Nash equilibria of the network game. In particular, note that players' actions in a Nash equilibrium of the network game are proportional to their actions at a Nash equilibrium of the module game.

The following example illustrates a network game and the corresponding nested games. **Example 1.** Consider the network with eleven players depicted in Figure 1. Clearly, the partition  $\mathbf{p} = \{M_1, M_2, M_3, M_4, M_5\}$ , where  $M_1 = \{1\}$ ,  $M_2 = \{2, 6\}$ ,  $M_3 = \{7, 11\}$ ,  $M_4 = \{3, 4, 5\}$ , and  $M_5 = \{8, 9, 10\}$ , is modular since each group is a module. Below, we depict the underlying networks of the various games.

- (1) The network game  $\Gamma(\mathbf{g}, \delta)$ .

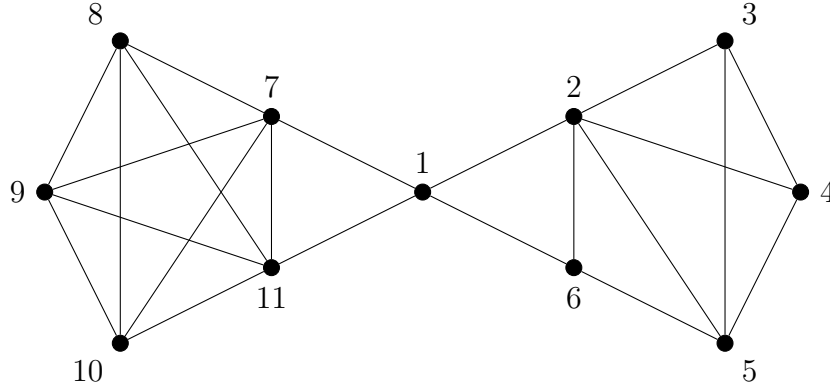


Figure 1: The network  $\mathbf{g}$ .

- (2) The nested games.
  - (a) The module game  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, 5$ .

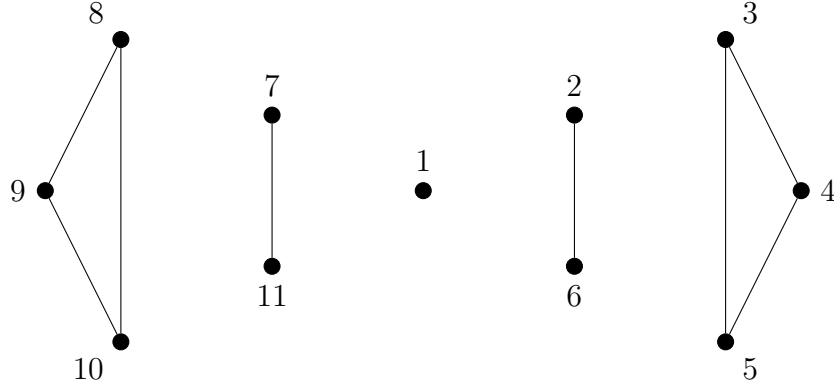


Figure 2: The module network  $\mathbf{g}_{M_k}$ , for each  $k = 1, \dots, 5$ .

(b) The composite game  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; y_{M_1}^*, \dots, y_{M_5}^*)$ .

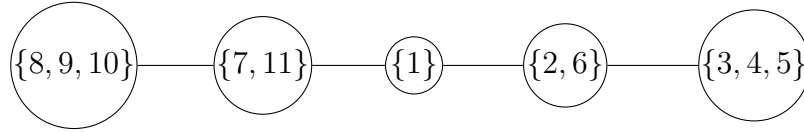


Figure 3: The quotient network  $\mathbf{g}/\mathbf{p}$ .

#### 4. HIERARCHICAL DECOMPOSITION

Now, we will further exploit the decomposition of the network game into nested games. More specifically, we will establish that the network game can be arranged into a unique hierarchy of nested games. Key to this is the concept of strong modules. A module  $M$  is called a strong module if, for any module  $M' \neq M$ , it holds that either  $M' \cap M = \emptyset$  or one module is included in the other. We say that a strong module  $M$  is a descendant of another strong module  $M'$  if  $M \subset M'$  and there is no other strong module  $M^*$  such that  $M \subset M^* \subset M'$ .

The descendant relation yields a tree on the set of strong modules, called the modular decomposition tree of the network, where the set of players  $\{1, \dots, n\}$  is the root, the single players  $\{1\}, \dots, \{n\}$  are the leaves, and any other strong module is an internal node. The nodes of the modular decomposition tree are labeled in three ways: *parallel* when the descendants are all non-neighbors of each other, *series* when the descendants are all neighbors of each other, and *prime* otherwise. The modular decomposition tree of a network is unique, as illustrated in the example in Figure 4.<sup>5</sup>

<sup>5</sup>In fact, the modular decomposition tree constitutes an exact alternative representation of the network whenever the structure of each prime module is depicted.

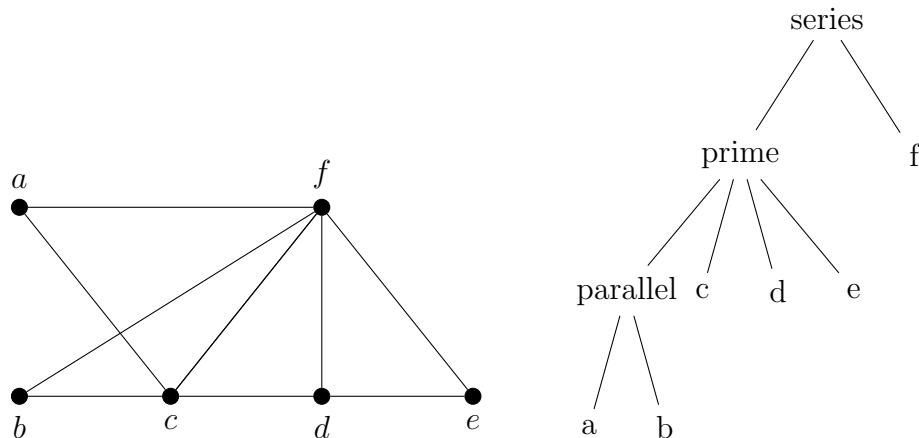


Figure 4: Modular decomposition tree of a network.

The following result relates a Nash equilibrium of a strong module game to the Nash equilibria of the nested games of the descendants' partition.

**Theorem 2.** *Given a strong module  $M$  with descendants' partition  $\mathbf{p}_M = (D_1, \dots, D_T)$ , the following are equivalent:*

- (1)  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M, \delta)$
- (2)  $\mathbf{x}^* = (r_1^* \mathbf{y}_{D_1}^*, \dots, r_T^* \mathbf{y}_{D_T}^*)$  such that
  - (i)  $\mathbf{y}_t^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{D_t}, \delta)$ , for each  $t = 1, \dots, T$ ,
  - (ii) If  $M$  is parallel, then for each  $t = 1, \dots, T$ , it holds that  $r_t^* = 1$ ,
  - (iii) If  $M$  is series, then for almost every  $\delta$ ,<sup>6</sup> either for each  $t \in A$  it holds that  $y_{D_t}^* > \frac{1}{\delta}$ , or for each  $t \in A = \{1, \dots, T\}$  it holds that  $y_{D_t}^* < \frac{1}{\delta}$ , and

$$r_t^* = \frac{\frac{1}{1 - \delta y_{D_t}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}}, \text{ and}$$

- (iv) If  $M$  is prime, then  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$ .

Theorem 2 provides insights into the strategic interactions between the representative players of descendants of a strong module. In interpretation, the representative players of descendants of a parallel module can be thought of as strategic complements, those of a series module can be thought of as strategic substitutes, and those of a prime module can be thought of as an intermediate case of strategic substitutes and strategic complements.

Given that the network game can be decomposed along the nodes of the modular decomposition tree into a unique hierarchy of nested games, Theorem 2 can be used to

<sup>6</sup>We say that a property holds for almost every  $\delta$  if it holds for every  $\delta$  except a finite number of values.

compute Nash equilibria using the bottom-up technique along the modular decomposition tree.

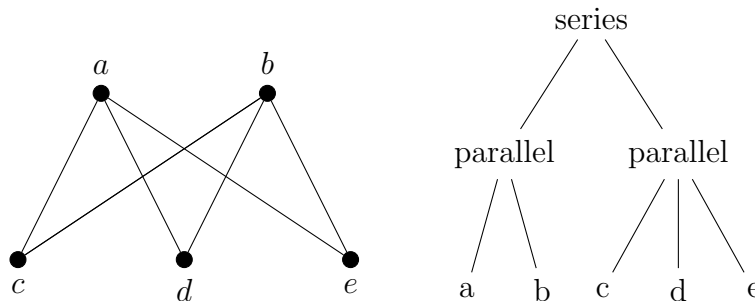


Figure 5: A cograph network.

In particular, for the special class of networks known as *cographs*, which consist of networks with only parallel and series modules in their modular decomposition tree,<sup>7</sup> it follows that, for almost every  $\delta$ , Nash equilibria can be computed immediately. The following example illustrates this point.

**Example 2.** Consider the cograph network with five players depicted in Figure 5. Observe that the entire set of players  $\{a, b, c, d, e\}$  is a series module with descendants  $D_1 = \{a, b\}$  and  $D_2 = \{c, d, e\}$ , which in turn are both parallel modules with only single players as descendants. Therefore, it holds that  $y_{D_1} = 2$  and  $y_{D_2} = 3$ , which constitute the thresholds of (2)(iii) in Theorem 2. Then using the computation in Theorem 2 bottom-up along the modular decomposition tree, we can compute all Nash equilibria as depicted in Figure 6.

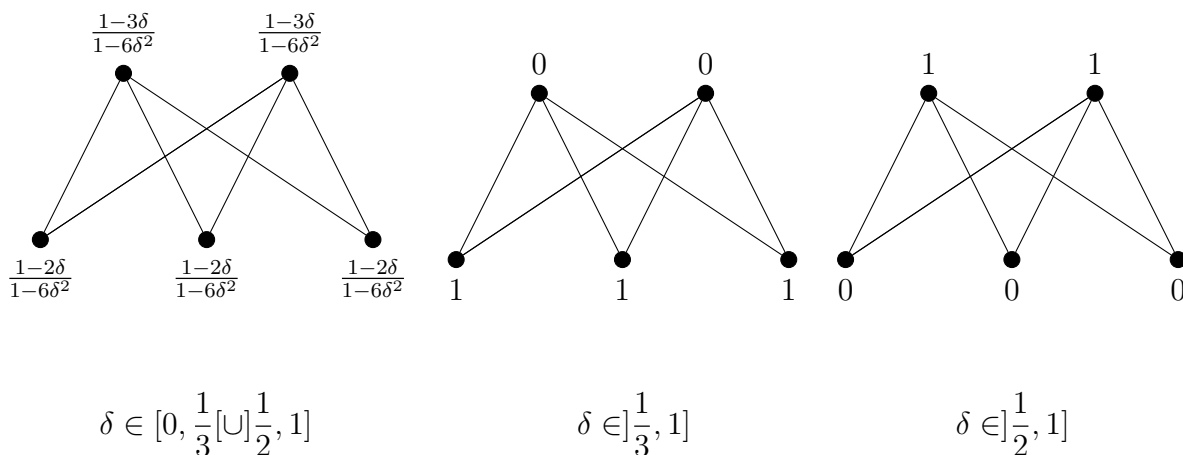


Figure 6: Nash equilibria of a cograph network.

<sup>7</sup>The class of cographs has been intensively studied since it was discovered independently by several authors in the 1970s.

Observe that neither an interior equilibrium nor a corner equilibrium exists for the whole range of  $\delta \in [0, 1]$ . Moreover, observe that at the interior equilibrium for (low)  $\delta \in [0, \frac{1}{3}[$  the aggregate action of the large representative player  $D_2$  is higher than that of the small representative player  $D_1$  and quite interestingly this gets reversed for (high)  $\delta \in ]\frac{1}{2}, 1]$ . The intuition is as follows. Recall the parameter  $\delta$  gives the substitutability between own and neighbors' actions. Therefore, for a low  $\delta$  players substitute other players' actions little, resulting in large representative players becoming more active, whereas for a high  $\delta$  players substitute other players' actions more, resulting in large representative players becoming less active.

## 5. AN APPLICATION: PUBLIC GOODS IN NETWORKS

Now, we provide an application of our results to the public goods in a network model, introduced in Bramoullé and Kranton (2007), which can be investigated as a  $\Gamma(\mathbf{g}, 1)$  game. Recall that for a profile of contributions to be a Nash equilibrium, it has to be the case that every player contributes nothing to the public good if the sum of his neighbors' contributions exceeds 1 or contributes exactly the difference between 1 and the sum of his neighbors' contributions. Therefore, at a Nash equilibrium, we may distinguish three types of players: *free-riders*, who contribute nothing; *experts*, who make full contributions; and the *others*. Bramoullé and Kranton (2007) insightfully show that specialized equilibria—that is, equilibria with only experts and free-riders—correspond to maximal independent sets of the network and therefore are always guaranteed to exist.

Specialized equilibria are of interest as they illustrate in an acute form how the network can lead to specialization. However, beyond specialized equilibria, very little is known about other equilibria such as distributed equilibria, where all players make positive contributions, and hybrid equilibria, which are neither specialized nor distributed. Distributed equilibria can be especially of interest given their normative importance, because all players share the burden of contributing to the public good, but they are not always guaranteed to exist. For instance, distributed equilibria are not possible in star networks. Moreover, even when distributed equilibria exist, very little is known about their properties beyond the symmetric contribution equilibrium in regular networks.

In the following, we will provide a condition on the modular decomposition of the network that is necessary for the existence of a distributed equilibrium. We say that a series module is *uncentered* if all (or none) of its descendants are single players. More

specifically, an uncentered series module rules out the possibility of having both a single player and a non-single player as descendants, which, as shown below, precludes the distributed equilibrium.

**Proposition 1.** *If a distributed equilibrium exists, then all series modules are uncentered.*

The intuition for the necessary condition of Proposition 1 can be explained as follows. In a distributed equilibrium, it must be the case that every player makes a strictly positive contribution. However, the (simultaneous) presence of a single player and a non-single player as descendants of a series module brings about a mismatch between what these players contribute and consume of the public goods, leading one of them to become a free-rider. The next result shows that the necessary condition becomes also sufficient for a special class of networks.

**Proposition 2.** *If the network is a cograph, then a distributed equilibrium exists if and only if all series modules are uncentered.*

It is worth noting that Sun (2012) also provides a sufficient and necessary condition for the existence of a distributed equilibrium for a general class of networks. Our analysis differs from Sun (2012) in at least two key aspects. First, it highlights the role intermediate network architectures play in determining public goods provision. Second, it provides an algorithm to compute the distributed equilibria for the special class of cographs. The following example illustrates these points.

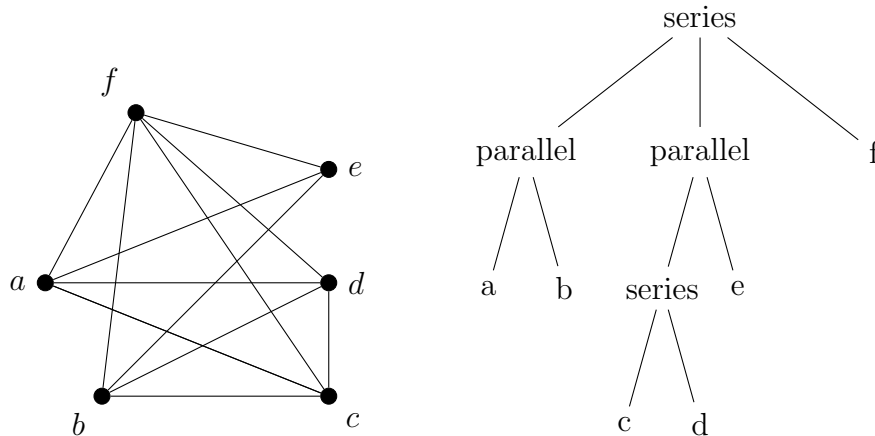


Figure 7: Cograph network with six players.

**Example 3.** Consider the cograph network with six players depicted in Figure 7. Then using the computation in Theorem 2 bottom-up along the modular decomposition tree, we

can compute all Nash equilibria as shown in Figure 8. Observe that there is no distributed equilibrium, which can be explained by the fact that the series module consisting of the entire set of players has both a single player and a non-single player as descendants.

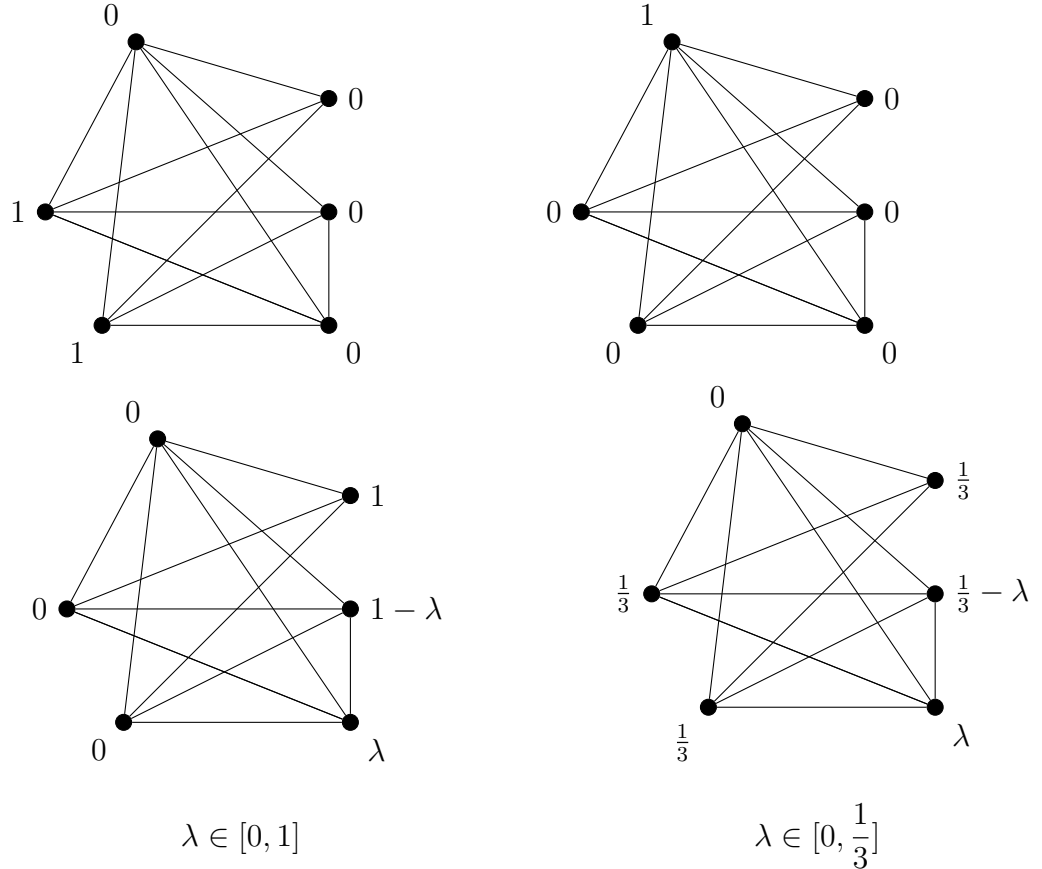


Figure 8: Nash equilibria of a cograph network.

## 6. AGGREGATION AND STABILITY

We now consider the issue of stability, which is often invoked to refine the set of Nash equilibria. In this respect, we closely follow Bramoullé, Kranton, and D’Amours (2014) and consider a myopic adjustment process defined, for each consumer  $i = 1, \dots, n$ , by

$$\dot{x}_i = f_i(\mathbf{x}_{-i}) - x_i,$$

where  $f_i(\mathbf{x}_{-i})$  is player  $i$ ’s best-reply function. The Nash equilibrium  $\mathbf{x}^*$  is “locally asymptotically stable” if there exists a neighborhood of  $\mathbf{x}^*$  such that if the above system starts at any point inside this neighborhood, it converges back to  $\mathbf{x}^*$ . In interpretation, stable equilibria are robust to small perturbations in players’ actions.<sup>8</sup>

<sup>8</sup>See Bervoets and Faure (2016) for a more recent contribution on stability in network games.

Before investigating stability, we first relax the uniformity of links between groups of players in a modular partition. Given a partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  of the set of players  $N$ , we now define the quotient network,  $\mathbf{g}/\mathbf{p}$ , in the following way: two disjoint groups of players  $M_k$  and  $M_h$  are linked in the quotient network if a player in  $M_k$  is adjacent to a player in  $M_h$ .

We say a profile of actions  $\mathbf{x} = (x_1, \dots, x_n)$  is *aggregate* with respect to the partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  if there exists  $\alpha \in [0, 1]$  such that for each  $k, i$  in  $M_k$ , and  $h \in \mathcal{N}_k(\mathbf{g}/\mathbf{p})$ , it holds that<sup>9</sup>

$$x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^* = \alpha x_{M_h}^*.$$

In interpretation, at an aggregate profile, players in each group  $M_h$  affect players in any linked group  $M_k$  in the same way—that is, by the same proportion of aggregate play.

**Theorem 3.** *Given an aggregate profile  $\mathbf{x}^*$  with respect to the partition  $\mathbf{p} = \{M_1, \dots, M_K\}$ , the following are equivalent:*

- (1)  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$
- (2)  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that
  - (a)  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, K$ , and
  - (b)  $\mathbf{r}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; \alpha y_{M_1}^*, \dots, \alpha y_{M_K}^*)$ .

Theorem 3 shows that an aggregate profile is a Nash equilibrium if and only if it is a combination of Nash equilibria of the nested games. For simplicity, from now on, we will call such a profile an *aggregate equilibrium*.

Note that if  $\mathbf{p} = \{M_1, \dots, M_K\}$  is a modular partition, then every Nash equilibrium is an aggregate equilibrium corresponding to  $\alpha = 1$ . The following example lists an aggregate equilibrium without the partition being modular.

**Example 4.** Consider the regular network with six players depicted in Figure 9. Clearly, the partition  $\mathbf{p} = \{M_1, M_2\}$ , where  $M_1 = \{1, 2, 3\}$  and  $M_2 = \{4, 5, 6\}$ , is not modular since neither  $M_1$  nor  $M_2$  is a module. Now consider the symmetric Nash equilibrium  $\mathbf{x}^* = \frac{1}{1+3\delta}(1, 1, 1, 1, 1, 1)$  of  $\Gamma(\mathbf{g}, \delta)$ . It can be easily checked that  $\mathbf{x}^*$  is aggregate with respect to the partition  $\mathbf{p} = \{M_1, M_2\}$  since, for each  $i$  in  $M_1$  and  $j$  in  $M_2$ , it holds that

$$x_{\mathcal{N}_i(\mathbf{g}_{M_2})}^* = x_{\mathcal{N}_j(\mathbf{g}_{M_1})}^* = \frac{1}{3}x_{M_1}^* = \frac{1}{3}x_{M_2}^* = \frac{1}{1+3\delta}.$$

<sup>9</sup>By abuse of notations,  $\mathcal{N}_i(\mathbf{g}_{M_h})$  denotes  $i$ 's neighbors in  $M_h$ .

Therefore,  $\mathbf{x}^*$  is an aggregate equilibrium. Moreover, observe that  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, r_2^* \mathbf{y}_{M_2}^*)$ , where for each  $k = 1, 2$ ,  $\mathbf{y}_{M_k}^* = \frac{1}{1+2\delta}(1, 1, 1)$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$  and  $\mathbf{r}^* = \frac{1+2\delta}{1+3\delta}(1, 1)$  is a Nash equilibrium of  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; \frac{1}{3} y_{M_1}^*, \frac{1}{3} y_{M_2}^*)$ .

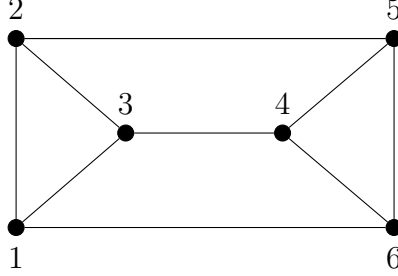


Figure 9: Regular network with six players.

The following result relates the stability of the aggregate equilibrium to the stability of the Nash equilibria of the nested games. For simplicity, we assume that all inactive players are strictly inactive. Let us also consider the set  $K'$  of groups that contain active players.

**Theorem 4.** *If the aggregate equilibrium  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  is stable, then the Nash equilibrium  $\mathbf{y}_{M_k}^*$ , for each  $k \in K'$ , and  $\mathbf{r}^*$  are stable.*

Theorem 4 shows that the stability of the aggregate equilibrium implies the stability of the Nash equilibria of the nested games. The opposite, however, is not true: the stability of the aggregate equilibrium cannot be deduced from the stability of the Nash equilibria of the nested games. The following example illustrates this point.

**Example 5.** Consider again the aggregate equilibrium  $\mathbf{x}^* = \frac{1}{1+3\delta}(1, 1, 1, 1, 1, 1)$  in Example 4. Since  $\lambda_{\min}(\mathbf{G}) = -2$ , it holds that  $\mathbf{x}^*$  is stable for  $\delta \in [0, \frac{1}{2}[$  and unstable for  $\delta \in [\frac{1}{2}, 1]$ . Note that, for each  $k = 1, 2$ ,  $\mathbf{y}_{M_k}^* = \frac{1}{1+2\delta}(1, 1, 1)$  is stable for  $\delta \in [0, 1[$  since  $\lambda_{\min}(\mathbf{G}_{M_k}) = -1$  and  $\mathbf{r}^* = \frac{1+2\delta}{1+3\delta}(1, 1)$  is stable for  $\delta \in [0, 1]$  since  $\lambda_{\min}(\mathbf{G}/\mathbf{p}) = -\frac{1}{1+2\delta} > -1$ . Therefore the stability range is larger for the Nash equilibria of the nested games than for the aggregate equilibrium. In the following, we depict the underlying network and the stability range of the various Nash equilibria.

(1) The aggregate equilibrium  $\mathbf{x}^* = \frac{1}{1+3\delta}(1, 1, 1, 1, 1, 1)$  is stable for  $0 \leq \delta < \frac{1}{2}$ .

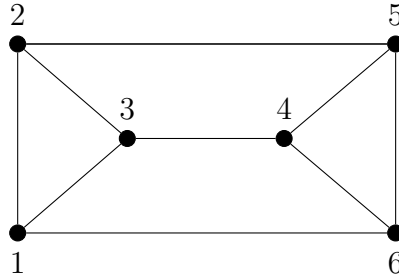


Figure 10: The network game.

(2) The Nash equilibria of the nested games  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, r_2^* \mathbf{y}_{M_2}^*)$ .

(a) The Nash equilibria  $\mathbf{y}_{M_1}^* = \mathbf{y}_{M_2}^* = \frac{1}{1+2\delta}(1, 1, 1)$  are stable for  $0 \leq \delta < 1$ .

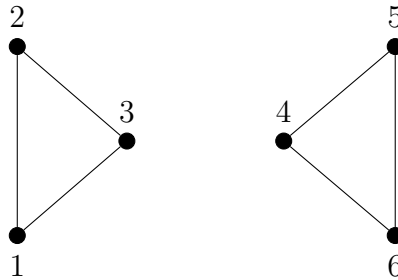


Figure 11: The group games.

(b) The Nash equilibrium  $\mathbf{r}^* = \frac{1+2\delta}{1+3\delta}(1, 1)$  is stable for  $0 \leq \delta \leq 1$ .

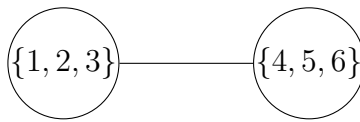


Figure 12: The composite game.

Our analysis shows that the stability of an aggregate equilibrium can be enhanced via the nested games.<sup>10</sup> Intuitively, there are fewer possible small perturbations of the Nash equilibria actions of the nested games than of the aggregate equilibrium actions. As a consequence, it may well be the case that small perturbations of the Nash equilibria actions of the nested games do not lead the system away from those equilibria, while (the larger set of) small perturbations of the aggregate equilibrium actions do lead the system away from equilibrium.

<sup>10</sup>Actually, it can be shown that the coarser is the partition of the set of players, the larger is the range of stability of the Nash equilibria of the nested games.

A key finding of Bramoullé, Kranton, and D’Amours (2014) is that the stability range of symmetric Nash equilibria, which have received much attention in the literature, is relatively small since, beyond a threshold, stable equilibria always involve some inactive players. Therefore, aggregation can be especially useful for enlarging the range of stability of symmetric equilibria.

## 7. CONCLUSION

Understanding, and making sense of, large economic networks is an increasingly important problem from an economic perspective, due to the ever-widening gap between technological advances in constructing such networks, and our ability to predict and estimate their properties. Throughout history, various concepts have been developed to reduce the inherent complexity found in large economic systems, thereby rendering them more amenable to economic analysis. One prominent example is aggregation, which aims to devise representative concepts that can be analyzed in a more tractable manner. For instance, a key question, which appeared in the seminal contributions of von Neumann and Morgenstern (1944), Chapter IX, Gorman (1953, 1961), and Shapley (1964, 1967), is: when does a group of individuals behave as if it were a single individual? In fact, amongst others, Shapley (1964) writes

*“An important question in the application of  $n$ -person game theory is the extent to which it is permissible to treat firms, committees, political parties, labor unions, nations, etc., as though they were individual players. Behind every game model played by such aggregates, there lies another, more detailed model: a compound game of which the original is the quotient. Given any solution concept, it is legitimate to ask how well it stands up under the aggregation -or disaggregation- of its players.”*

Our investigation of aggregation in network games is quite similar in motivation. Often, the reason such an argument holds in the above literature appears to hinge on having identical preferences or compositions.<sup>11</sup> Our approach suggests that aggregation holds for a similar reason in network games; however, the homogeneity is brought about by the network architecture rather than behavior or structure.

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<sup>11</sup>See Cornes and Hartley (2007) for a more recent contribution on aggregation.

Our findings could potentially have empirical applications to many network models in economics, including public goods and targeting/finding the key players policies. Nonetheless, it remains to be seen whether other approaches from the vast and important literature on network position similarity, across myriad disciplines, ranging from biology and sociology to computer science—see, for example, Gagneur et al. (2004) and Newman (2006)—could be useful to further analyze complex strategic interactions.

## 8. APPENDIX

**Proof of Theorem 1.** First, observe that a profile of actions  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if for each player  $i \in N$

$$x_i^* = \begin{cases} 1 - \delta x_{\mathcal{N}_i(\mathbf{g})}^* & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g})}^* \leq 1 \\ 0 & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g})}^* > 1. \end{cases} \quad (8.1)$$

Since  $M_k$  is a module, for each  $i \in M_k$  and for each  $h \neq k$ , it holds that the set of neighbors of  $i$  in  $M_h$ , that is,  $\mathcal{N}_i(\mathbf{g}_{M_h})$ , is independent of the choice of  $i \in M_k$ . Let us posit

$$r_k^* \stackrel{\text{def}}{=} \max\left\{1 - \delta \sum_{h \in \mathcal{N}_k(\mathbf{g}/\mathbf{p})} x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^*, 0\right\}.$$

Then, since for each  $i \in M_k$

$$\mathcal{N}_i(\mathbf{g}) = \bigcup_{h \in k \cup \mathcal{N}_k(\mathbf{g}/\mathbf{p})} \mathcal{N}_i(\mathbf{g}_{M_h}),$$

it holds that

$$\delta x_{\mathcal{N}_i(\mathbf{g})}^* = \delta \sum_{h \in k \cup \mathcal{N}_k(\mathbf{g}/\mathbf{p})} x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^* = \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* + r_k^*. \quad (8.2)$$

Also let

$$\mathbf{y}_{M_k}^* \stackrel{\text{def}}{=} \begin{cases} \frac{\mathbf{x}_{M_k}^*}{r_k^*} & \text{if } r_k^* > 0 \\ \text{a Nash equilibrium of } \Gamma(\mathbf{g}_{M_k}, \delta) & \text{if } r_k^* = 0. \end{cases}$$

Hence, in view of (8.1) and (8.2),  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if for each module  $k = 1, \dots, K$

$$r_k^* = \max\left\{1 - \delta \sum_{h \in \mathcal{N}_k(\mathbf{g}/\mathbf{p})} y_{\mathcal{N}_i(\mathbf{g}_{M_h})}^* r_h^*, 0\right\}$$

and for each player  $i \in M_k$  it holds that

$$x_i^* = \begin{cases} r_k^* - \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* \leq r_k^* \\ 0 & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* > r_k^* \end{cases}$$

or, equivalently,

$$y_i^* = \begin{cases} 1 - \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* & \text{if } \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* \leq 1 \\ 0 & \text{if } \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* > 1. \end{cases}$$

Therefore,  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that  $\mathbf{r}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}/\mathbf{p}, \delta; y_{M_1}^*, \dots, y_{M_K}^*)$  and  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, K$ .  $\square$

**Proof of Theorem 2.** From Theorem 1, it holds that  $\mathbf{x}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M, \delta)$  if and only if  $\mathbf{x}_M^* = (r_1^* \mathbf{y}_{D_1}^*, \dots, r_T^* \mathbf{y}_{D_T}^*)$  such that  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M/\mathbf{p}_M, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  and  $\mathbf{y}_{D_t}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{D_t}, \delta)$ , for each  $t = 1, \dots, T$ .

If  $M$  is prime, then the equivalence follows from the result above.

If  $M$  is parallel, then  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M/\mathbf{p}_M, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  is equivalent to

$$r_t^* = 1 \text{ for each } t = 1, \dots, T,$$

since  $\mathcal{N}_t(\mathbf{g}_M/\mathbf{p}_M) = \emptyset$  for each  $t = 1, \dots, T$ .

If  $M$  is series, then  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M/\mathbf{p}_M, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  is equivalent to

$$r_t^* = 1 - \delta \sum_{s \in A \setminus \{t\}} y_{D_s}^* r_s^* \text{ for each } t \in A \quad (8.3)$$

and

$$\delta \sum_{s \in A} y_{D_s}^* r_s^* \geq 1 \text{ if } A \neq \{1, \dots, T\}. \quad (8.4)$$

Let

$$\mathbf{v} \stackrel{\text{def}}{=} \left( \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} \right)_{s \in A} \text{ and } \mathbf{U} \stackrel{\text{def}}{=} \text{diag}(1 - \delta y_{D_s}^*)_{s \in A}.$$

Then (8.3) is equivalent to

$$(\mathbf{I} + \mathbf{1}\mathbf{v}^T)\mathbf{U}\mathbf{r}_A^* = \mathbf{1}.$$

From the Sherman–Morrison formula, provided that  $1 + \mathbf{v}^T\mathbf{1} \neq 0$ , it holds that

$$\mathbf{r}_A^* = \mathbf{U}^{-1}(\mathbf{I} + \mathbf{1}\mathbf{v}^T)^{-1}\mathbf{1} = \mathbf{U}^{-1}\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{v}^T}{1 + \mathbf{v}^T\mathbf{1}}\right)\mathbf{1} = \mathbf{U}^{-1}\left(\mathbf{1} - \frac{\mathbf{v}^T\mathbf{1}}{1 + \mathbf{v}^T\mathbf{1}}\mathbf{1}\right) = \frac{1}{1 + \mathbf{v}^T\mathbf{1}} \mathbf{U}^{-1}\mathbf{1}.$$

Hence, for each  $t \in A$ , it holds that

$$r_t^* = \frac{\frac{1}{1-\delta y_{D_t}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}}.$$

Note that since  $r_t^* > 0$  for each  $t \in A$ , it follows from above that either  $y_{D_t}^* > \frac{1}{\delta}$  for each  $t \in A$  or  $y_{D_t}^* < \frac{1}{\delta}$  for each  $t \in A$ . Moreover, in view of (8.4), if  $A \neq \{1, \dots, T\}$  then

$$\frac{\sum_{s \in A} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}} = 1 - \frac{1}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}} \geq 1,$$

which implies that

$$\sum_{s \in A} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} < -1.$$

Hence if  $A \neq \{1, \dots, T\}$ , then it holds that  $y_{D_t}^* > \frac{1}{\delta}$  for each  $t \in A$ .

Conversely, it is easy to check that if either for each  $t \in A$  it holds that  $y_{D_t}^* > \frac{1}{\delta}$ , or for each  $t \in A = \{1, \dots, T\}$  it holds that  $y_{D_t}^* < \frac{1}{\delta}$ , and

$$r_t^* = \frac{\frac{1}{1-\delta y_{D_t}^*}}{1 + \sum_{s \in A} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}},$$

then  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M/\mathbf{p}_M, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$ .  $\square$

**Proof of Proposition 1.** Let  $\mathbf{x}^*$  be a Nash equilibrium of  $\Gamma(\mathbf{g}, 1)$  such that  $x_i^* > 0$ , for each  $i \in N$ . Let  $M$  be a series module. From Theorem 1, there exists a real number  $r_M > 0$  such that  $\mathbf{x}^* = r_M \mathbf{y}_M^*$ , where  $\mathbf{y}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M, 1)$ . Suppose that  $M$  is not uncentered. Let  $\mathbf{p}_M = (D_1, \dots, D_T)$  denote the descendants' partition of  $M$ . Then, there exists  $1 \leq t_1 \neq t_2 \leq T$  such that  $D_{t_1} = \{i_1\}$  is a single player and  $D_{t_2}$  is not a single player. Note that each player in  $D_{t_2}$  is not connected to all other players in  $D_{t_2}$ . Otherwise,  $D_{t_2}$  is not a direct descendant of  $M$ .

At the Nash equilibrium  $\mathbf{y}_M^*$ , each player's action is a best reply to his neighbors' actions. In particular, it holds for player  $i_1$  that

$$y_{i_1}^* + \sum_{i \in D_{t_2}} y_i^* + \sum_{t \neq t_1, t_2} y_{D_t}^* = 1$$

and for a player  $i_2 \in D_{t_2}$  that

$$y_{i_2}^* + \sum_{i \in \mathcal{N}_{i_2}(\mathbf{g}_M) \cap D_{t_2}} y_i^* + y_{i_1}^* + \sum_{t \neq t_1, t_2} y_{D_t}^* = 1,$$

which together imply

$$\sum_{i \in \{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g}_M)\}^c \cap D_{t_2}} y_i^* = 0.$$

This is a contradiction since  $\{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g}_M)\}^c \cap D_{t_2} \neq \emptyset$  and  $y_i^* > 0$ , for each  $i \in M$ .  $\square$

**Proof of Proposition 2.** Suppose the network  $\mathbf{g}$  is a cograph. Therefore, the network  $\mathbf{g}$  has only parallel and series modules in its modular decomposition tree. If all series module are uncentered, then, given a series module  $M$ , with direct descendants' partition  $\mathbf{p}_M = (D_1, \dots, D_T)$ , either all or none of the direct descendants are single players. If all direct descendants are single players, then the symmetric contribution  $\frac{1}{T+1}$  is a Nash equilibrium of  $\mathbf{\Gamma}(\mathbf{g}_M, 1)$ . If none of  $M$ 's direct descendants is a single player, then for each  $t = 1, \dots, T$  and for any Nash equilibrium  $\mathbf{y}_t^*$  of  $\mathbf{\Gamma}(\mathbf{g}_{D_t}, \delta)$ , it holds that  $y_{D_t}^* \geq 2$  since  $D_t$  is a parallel module with at least two direct descendants. From (2)(iii) in Theorem 2, it follows that there exists a Nash equilibrium of the composite game such that  $r_t^* > 0$  for each  $t = 1, \dots, T$ . Therefore one can use (2)(ii) and (3)(iii) in Theorem 2 recursively along the nodes of the modular decomposition tree in order to construct a distributed equilibrium.  $\square$

**Proof of Theorem 3.** The proof is similar to the proof of Theorem 1.  $\square$

**Proof of Theorem 4.** The proof relies on the Brouwer and Haemers (2011) version of the generalized interlacing eigenvalue theorem, as stated below.

**Theorem.** (*Brouwer and Haemers*). *Let  $\mathbf{S}$  be a real  $n \times m$  matrix such that  $\mathbf{S}^T \mathbf{S} = \mathbf{I}$ . Let  $\mathbf{K}$  be a real symmetric matrix of order  $n$ . Define  $\mathbf{J} = \mathbf{S}^T \mathbf{K} \mathbf{S}$ . Then the eigenvalues of  $\mathbf{J}$  interlace those of  $\mathbf{K}$ .*

Note that the classical interlacing eigenvalue theorem holds as a special case of the generalized interlacing eigenvalue theorem if one takes  $\mathbf{S} = [\mathbf{I}, \mathbf{0}]$ . For each  $k \in K'$ , let  $M'_k = M_k \cap A$  and  $\mathbf{p}' = \{M'_k\}_{k \in K'}$ . The aggregate equilibrium  $\mathbf{x}^*$  is stable if and only if all eigenvalues of the matrix  $\mathbf{I} + \delta \mathbf{G}_A$  have positive real parts, which is equivalent to  $\lambda_{\min}(\mathbf{G}_A) > -\frac{1}{\delta}$  since all the eigenvalues of  $\mathbf{G}_A$  are real.

First observe that from the classical interlacing eigenvalue theorem it follows that  $\lambda_{\min}(\mathbf{G}_{M'_k}) \geq \lambda_{\min}(\mathbf{G}_A) > -\frac{1}{\delta}$ , for each  $k \in K'$ . Therefore  $\mathbf{y}_{M'_k}^*$  is stable, for each  $k \in K'$ .

Second observe that rows and columns of  $\mathbf{G}_A$  can be partitioned as

$$\mathbf{G}_A = (\mathbf{G}_{kh})_{k,h \in (K')^2},$$

where  $\mathbf{G}_{kh}$  lists the links connecting players in  $M_k$  to players in  $M_h$ .<sup>12</sup> Let  $\mathbf{S}$  be the  $A \times K'$  matrix defined as follows:

$$s_{ik} \stackrel{\text{def}}{=} \begin{cases} \frac{y_i}{\|\mathbf{y}_{M'_k}\|} & \text{if } i \in M'_k, \\ 0 & \text{otherwise} \end{cases},$$

$$\mathbf{U} = \text{diag}\left(\frac{\|\mathbf{y}_{M'_k}\|}{\sqrt{y_{M'_k}}}\right)_{k \in K'},$$

and

$$\mathbf{V} = \text{diag}(\sqrt{y_{M'_k}})_{k \in K'}.$$

Since  $(\mathbf{I} + \delta \mathbf{G}_{kk})\mathbf{y}_{M'_k} = \mathbf{1}$  and  $\mathbf{G}_{kh}\mathbf{y}_{M'_k} = \alpha y_{M'_k} \mathbf{1}$ , it follows that

$$\mathbf{V}\mathbf{U}\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{S}\mathbf{U}\mathbf{V}^{-1} = \mathbf{I} + \delta \mathbf{G}_A/\mathbf{p}',$$

where  $\mathbf{G}_A/\mathbf{p}'$  denotes the adjacencies of active groups in the Nash equilibrium  $\mathbf{r}^*$ .

Observe that  $\mathbf{S}^T\mathbf{S} = \mathbf{I}$ . Hence, it follows from the generalized interlacing eigenvalue theorem that the eigenvalues of  $\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{S}$  are positive since they interlace those of  $\mathbf{I} + \delta \mathbf{G}_A$ . From the sharp bounds provided by Ostrowski (1959), it holds that the eigenvalues of the symmetric matrix  $\mathbf{U}\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{S}\mathbf{U}$  are also positive since they are given by  $\psi_i \lambda_i$ , where  $\lambda_i$  is an eigenvalue of  $\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{S}$  and  $\psi_i$  lies between the smallest and the largest eigenvalues of  $\mathbf{U}$ . Hence, the matrix  $\mathbf{I} + \delta \mathbf{G}_A/\mathbf{p}'$  also has positive eigenvalues, being similar to  $\mathbf{U}\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G}_A)\mathbf{S}\mathbf{U}$ . Therefore,  $\mathbf{r}^*$  is a stable Nash equilibrium.  $\square$

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<sup>12</sup>Observe that  $\mathbf{G}_{kk} = \mathbf{G}_{M_k}$ .

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