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Production and Endogenous Bankruptcy under Collateral Constraints

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Production and Endogenous Bankruptcy under Collateral Constraints*

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Abstract

We propose an extension of the incomplete markets general equilibrium model with production to situations in which firms default. In the model, firms are assumed to be owned by a single individual whose roles as entrepreneur and consumer are anonymous. Assets are exogenously collateralised and are seized together with future output in case of defaulting. Default is thus endogenous as output determines the deliveries from the assets issued. In turn, the receipts from each asset purchased are assumed to depend on a fraction of the firm's output. This fraction is assumed to be anonymous and is treated as given by individuals who will anticipate it, in equilibrium, as a fraction of the whole output of the economy, generating counter-cyclical default. We show that, under usual assumptions on utilities and technologies, an equilibrium always exists. We then discuss the implications of this setting for Fisher's "separation theorem" and the irrelevance of the financial policies of the firm.

Keywords: General equilibrium; Incomplete markets; default; production economy.
JEL Codes: D52; C81.

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Non-technical summary

The cyclical behaviour of firm bankruptcy displays a clear counter-cyclical pattern: firms tend to go bankrupt in recessions. This leads to an amplification of business cycles and has important consequences to understand asset prices and the evolution of other macro variables like employment. However, macroeconomic models in which firms are allowed to go out of business are scarce. In this paper, we take a step in the direction of understanding the general equilibrium effects of default and bankruptcy.

We build a general equilibrium model with many firms and households where agents live for two periods. There is uncertainty about the future state of the economy, but asset markets are incomplete, in the sense that there is not a complete set of contracts to protect lenders and borrowers from future contingencies. Firms are assumed to be owned by a single individual whose roles as both consumer and entrepreneur are anonymous. I.e. the assets they issue can be used to consume or produce, but the lender cannot distinguish between these two activities when buying assets. Assets are exogenously collateralised by the household's capital and, because entrepreneurs are assumed to have unlimited liability, they are seized together with output in case the firm defaults on its future promises to deliver the principal and a return. This dependence on second period output leads to endogenous default. It is important to note that, in models without production, the existence of collateral provides insurance for lenders. However, with production, problems of asymmetric information reappear because deliveries from borrowers depend on the level of production. To solve these asymmetric information issues, we make receipts from assets purchased anonymous by assuming a type of *tranching*: the production (and collateral) of a consumer/firm backs all the assets they issue. Because this anonymity, lenders have to anticipate a default rate on their portfolio that will depend on the average level of production of firms in the economy. This will depend on the expected state of the economy and, hence, aggregate economic activity (i.e. default is counter-cyclical).

We then use this model to analyse 3 questions. 1) Is there an equilibrium in which all markets clear and borrowers and lenders can achieve their optimal plans? This is important because, without an equilibrium, it would be impossible to determine prices and quantities and, hence, the impact of any exogenous changes like policies. 2) Does it matter whether firms finance their investment with their own funds or external funds? This is a key concept in corporate finance, because it relates to the possible real effects of the financial structure of firms (the so-called Modigliani-Miller Theorem). 3) Can we separate the firm's decisions from the objectives of the households (owners) as consumers? This matters because, if we can, then firms would exploit the best market opportunities regardless of the preferences of their owners (the so-called Fisher Separation Theorem). The answers to these three questions are: yes, no, and no, respectively.

1 Introduction

One of the key features of production activities is that they take time. In order to produce, firms have to carry out investments that are funded by either equity or debt. If there are financial frictions in the economy, this then leads to a close link between finance and the level of production of firms and, therefore, real economic activity. The impact of financial variables on the real economy has been the subject of an increasingly large literature in macroeconomics. This is especially so since the advent of the Global Financial Crisis (see Quadrini, 2011 and Brunnermeier et al. 2012 for overviews). Although that particular crisis originated mainly in sub-prime lending for housing market investment, business default or bankruptcy can also lead to large disruptions in economic activity and to the amplification of shocks. Figure 1 shows the evolution of business bankruptcy and delinquency rates in the US, which displays a clear counter-cyclical pattern. Both measures increase substantially during recession periods.[1] It is thus important to develop a theoretical framework which puts business default and bankruptcy at center stage in an economy with incomplete financial markets.

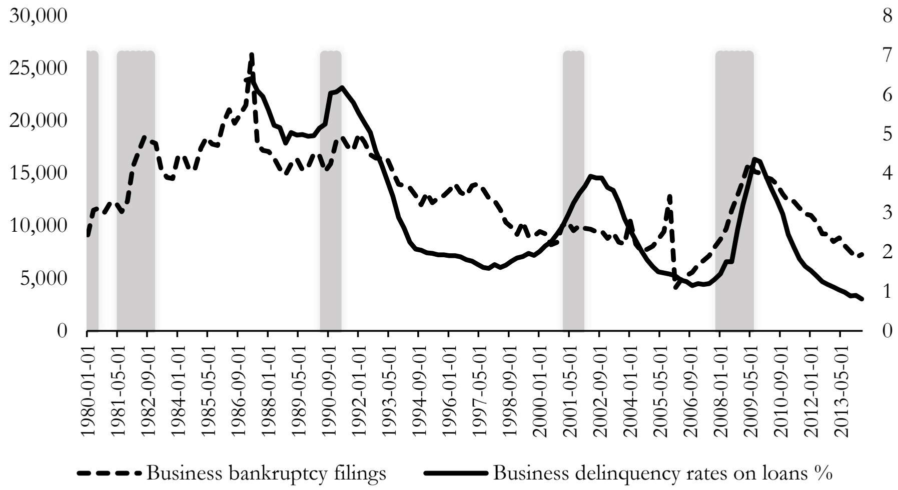

Figure 1: Business bankruptcy filings (left scale) and business delinquency rates (in percentage) on commercial banks (right scale) in the US, 1980-2014. Shaded areas are NBER recession dates.

Our objective is to take a step in this direction by proposing an extension of the incomplete markets general equilibrium (GEI) model with production to situations in

[1]There was a surge and then a drop in bankruptcy filings between 2005 and 2006 prompted by the October 2005 implementation of the Bankruptcy Abuse Prevention and Consumer Protection Act.

which producers can default. Our contribution is to establish a theoretical framework stressing the role of the production sector, which is absent in earlier studies with default and collateral, and to show that default and bankruptcy are consistent with the orderly functioning of markets.[2] That is to say, an equilibrium where firms default exists, although bankruptcy is a disequilibrium signal.

Our model is an incomplete markets multi-good multi-agent economy with two periods. Firms are assumed to be owned by a single individual[3] whose roles as both consumer and entrepreneur are anonymous. I.e. the assets they issue can be used to consume or produce, but the lender cannot distinguish between these two activities when buying assets. Assets are exogenously collateralised and, because entrepreneurs are assumed to have unlimited liability, they are seized together with output in case the firm defaults on its promises. This dependence on second period output makes deliveries endogenous, leading to endogenous default. It is important to note that, in models without production, the existence of collateral provides insurance for lenders. However, with production, problems of asymmetric information reappear because deliveries depend on the level of production. To solve this issue, we make receipts from assets purchased anonymous by assuming a type of tranching: the production (and collateral) of a consumer/firm backs all the assets they issue. Because this anonymity, lenders have to anticipate a default rate on their portfolio that will depend on the average level of production of firms in the economy. This will depend on the state of nature and, hence, aggregate economic activity (i.e. default is counter-cyclical). In order to prevent a zero production equilibrium, we also introduce an insurance mechanism issued by lenders that is equivalent to a discount on the price of the assets purchased. This mechanism follows Araujo et al. (2000), where the result is that borrowers are encouraged to offer more collateral.

Under usual assumptions on utilities and technologies, we prove the existence of equilibrium for a multi-good production economy where firms default on their promises which are assumed to be collateralised by durable goods as in Geanakoplos and Zame (2014). In addition to showing the existence of an equilibrium, we also analyse its non-triviality.

Our second goal is to consider the finance economy derived from the corresponding multi-good economy in order to analyse the financial policies of the firm under

[2]We clarify below the distinction between default and bankruptcy, which relates to the type of equilibrium analyzed in different parts of the paper.

[3]Our analysis will be limited to the case of sole proprietorship and not to firms with open capital. The latter turns out to be more complicated in presence of incomplete markets even without bankruptcy, see DeMarzo (1988).

bankruptcy and collateral constraints. More precisely, we study under which conditions Fisher's Separation and Modigliani-Miller Theorems hold in this sole proprietorship setting. Of course, we should not expect these theorems to hold in our setting because of the nonlinearity or nonadditivity of deliveries[4] in relation to short positions. In spite of that, it is still important to study the conditions leading to their failure. This is because the validity of these theorems is not only important from a perspective of corporate finance but also from a macroeconomic point of view (see Bisin, Gottardi and Ruta, 2009 and Gerbach, Haller and Muler, 2015).

To analyse the financial policies of the firm, we consider the finance economy associated to the multi-goods economy. Two kinds of equilibria are defined. The first one is the reduced-form equilibrium in which consumption and production activities are carried out by one individual whose roles as consumer and entrepreneur are indistinguishable. In this kind of setting the situation in which the individuals do not honor their financial commitments is called "default" because the financial decisions as consumers and firms are not separated. The second kind of equilibrium is the extensive-form, where the financial accounts of firms are separated from those of consumers, who are the owners of the firms. In this case, we call "bankruptcy" the situation in which firms do not honor their debts. For the latter definition to be well defined, we slightly modify the classical definition of the reduced-form equilibrium which requires the decomposition of the debt portfolio. Once this is done, we define the extensive-form equilibrium requiring that both consumers and entrepreneurs (firms) decide separately their consumption and production activities as well as their financial decisions in order to maximise their objective problems: consumers maximise their utility functions subject to their budget constraints and firms maximise the present value of their dividends.

Lastly, under a hypothesis called separation of deliveries (Lemma 3), we analyse the equivalence between the reduced-form equilibrium and the extensive-form equilibrium. From this equivalence we analyze the Modigliani-Miller Theorem and Fisher's Separation Theorem. The latter theorem is studied when the firm maximizes profits instead the present value of its dividends.

Our paper is related to several streams of the literature on GEI models. The analysis of existence and non-triviality was first addressed in GEI models with default penalties and without production by Dubey et al. (1990, 2005) and Araujo et al. (1998), and

[4]Deliveries in models with collateral are defined to be the minimum between the claim and the value of the collateral. The latter is non additive with respect to short sales, since the collateral includes the production level of the firm, which is independent of the sale of assets.

in an exogenous collateral setting without production by Steinert and Torres-Martínez (2007). Our concept of non-triviality follows Araujo et al. (1998) where the payment rate of economy is defined to depend indirectly on aggregate output. Our existence argument follows Araujo et al. (2000) who proposed an endogenous collateral model with a continuum of agents but without production. In that model, borrowers choose the collateral level guaranteeing their debts and lenders offer default-insurance contracts. In our model, borrowers do not choose any collateral level (i.e. the collateral is exogenous), and a continuum of agents is not required because the budget sets are always convex, although deliveries are endogenous. The endogeneity of deliveries is due to the fact that they depend on the production level. In case of default, borrowers will deliver the depreciated collateral plus the production level. We thus maintain the default-insurance contracts to obtain the market clearing condition in the truncated economy, which will be a fundamental aspect to obtain an equilibrium for the limit economy.

The Fisher Separation Theorem in a model with incomplete markets was established by Magill and Quinzii (1996) in a sole proprietorship setting without default or bankruptcy. More recently, Carvalho, Divino and Orrillo (2007) extended the Magill and Quinzii (1996) result to a situation where agents are allowed not to honor their financial commitments. They obtain their result by assuming, as in Dubey, Geanakoplos and Shubik (2005), that agents are punished in proportion to their default. That is, the difference between the claim and the delivery which is assumed to be a decision variable. Consequently, the decision on whether or not to default depends on the comparison between the desire for consuming and the utility penalty suffered if they default. Although this result involved default, it is only true provided that agents as entrepreneurs do not default regardless of whether or not consumers default. The Modigliani-Miller Theorem has been analyzed before in a bankruptcy setting with incomplete markets by Hellwig (1981). Hellwig (1981) was able to prove the theorem but only under very stringent conditions on the kind of securities traded by firms, essentially making the market structure equivalent to having assumed complete markets.[5]

The paper is organised as follows. Section 2 presents the model. In Section 3, we establish the existence and non-triviality of the equilibrium. In Section 4, financial policies are analysed in a sole proprietorship setting. Finally, Section 5 concludes.

[5]See also Gottardi and Kubler (2015) for an analysis of complete markets in a collateral setting like ours but without production.

2 The Model

The model has two periods, $t = 0, 1$. In the first period there is no uncertainty whereas in the second one there is. To avoid notational clutter, we will use the same symbol to denote both sets and their cardinal. The uncertainty is over S possible states of nature, $S = \{1, 2, \ldots, S\}$. In the economy there are L goods in both periods, and since in second period we have S states of nature, the commodity space will be $R^{L(S+1)}$. The price system of commodities is the vector $p = (p_0, \tilde{p})$ belonging to $R_+^{L(S+1)}$ where $\tilde{p} = (p_1, \ldots, p_s) \in R_+^{(LS)}$. This notation is extended to other variables throughout the paper.

There are F firms and H consumers. We assume that the number of consumers is greater than or equal to the number of firms. That is, $H \geq F$.

Each agent $h \in H$ is characterised by a utility function $U^h : R_+^{L(S+1)} \to R$ and an initial endowment vector of commodities $\omega^h \in R_+^{L(S+1)}$. Each firm f is characterised by their production set $Y^f \subset R^{L(S+1)}$. The elements of Y^f are called projects available to the firm and are denoted by $y = (y_o, \tilde{y}) \in Y^f \subset R^{L(S+1)}$.

2.1 Ownership and financial structures

The ownership structure is characterized by a function $h : F \to H$. This assigns each firm f to its single owner $h(f)$ who is assumed to be an entrepreneur with unlimited liability. We can also consider the function $f : H \to L$ which assigns a single firm $f(h)$ to each consumer h. Since $H \geq F$, there could be consumers who own no firms.

There are J real assets in the first period available for trading. Each asset j is sold at price π_j in the first period and promises a bundle A_s^j if state of nature s is revealed in the second period. These assets are collateralised as in Geanakoplos and Zame (2014). Thus, for each unit sold of asset j, it is necessary to purchase a bundle $C_j \in R_+^L$ backing its promise. The collateral is assumed to depreciate according to function $K_s : R_+^L \to R_+^L$ which depends on the state of nature.

2.2 Deliveries and receipts

Since we are assuming no separation between consumption and production activities, then there is no separation either between financial decisions of consumers and entrepreneurs. In the event of default, individuals will give up, in addition to physical collateral, their production. This is because of the unlimited liability assumption. Hence, for every state s, the delivery or repayment of borrowers for each portfolio $\varphi \in R^J_+$ sold will be given by:

$$D_s(\varphi, y_s) = \sum_{j \in J} D^j_s(\varphi_j, y_s),$$

where $D^j_s(\varphi_j, y_s)$ is defined to be $\min\{p_s A^j_s \varphi_j, p_s(y_s + K_s C_j \varphi_j)\}$.

Notice that the value of production y_s at price p_s is backing all the J assets. That is, we are considering a type of *tranching.* Consequently, lenders, in purchasing unit of the asset j, will only expect to receive in case of default: a) a fraction $\overline{y}^j_s$ of the total production of the firm measured in nominal terms plus b) the value of the depreciated collateral which backed the issuance of asset j. That fraction, which is a pro rata of all its different sellers' production, is exogenous for lenders but will be determined in equilibrium as a proportion of the aggregate output of the economy. Thus, the receipt for each unit of asset j purchased will be:

$$R^j_s = \min\{p_s A^j_s, \overline{y}^j_s + p_s K_s C_j\}.$$

In other words, lenders who suffer default will receive the value of depreciated collateral plus the average output level $\overline{y}^j_s$. For each portfolio purchased $\theta \in R^J_+$, its receipts at state s, denoted by $R_s\theta$, is defined as $\sum_{j\in J} R^j_s \theta_j$.

Remark 1: In the particular case in which the agents do not own any firms, so that $y_s = 0$ and therefore $\overline{y}^j_s = 0$, deliveries and receipts will reduce to the same expression as in the exogenous collateral model of Geanakoplos and Zame (2014), where deliveries and receipts at sate s for each unit of security traded j is $\min\{p_s A^j_s, p_s K_s C_j\}$.

2.3 Default insurance

When a borrower defaults on asset j in state s, they deliver $D_s(\varphi_j, y_s) = p_s y_s + p_s K_s C_j \varphi_j$. Hence, the amount of default becomes $(p_s A^j_s \varphi_j - (p_s y_s + p_s K_s C_j \varphi_j))$. Similarly, the investor (lender) who suffers default on asset j in state s will receive $R^j_s = \overline{y}^j_s + p_s K_s C_j$. Thus, the amount of this default, called default suffered, is $(p_s A^j_s - (\overline{y}^j_s + p_s K_s C_j))\,\theta_j$ for each amount θ_j of asset purchased j.

The problem is that borrowers will want their default to be as large as possible. This would occur if they, as entrepreneurs, do not produce. Knowing this, lenders will not have any incentive to invest as they would only receive the future value of depreciated collateral $p_s K_s C_j \theta_j$ in case of default. To prevent this zero production and lending equilibrium, we need a mechanism to incentivise entrepreneurs to increase production.

In what follows, we describe such mechanism and, then, in the next section, look at how this mechanism impacts the decisions of agents.

2.3.1 The default insurance market

In the absence of intermediaries or insurance companies to protect lenders from an eventual default, it will be them who sell default insurance contracts to borrowers.[6]

Suppose that there are S state-contingent default-insurance contracts traded in a default insurance market. Each agent, who trades a portfolio $(\theta, \varphi) \in R^J_+ \times R^J_+$, will have to add to the portfolio cost $\pi(\theta - \varphi)$ the following overall insurance net premium:

$$\sum_{s\in S} \beta_s \sum_{j\in J} \left[\left(p_s A^j_s \varphi_j - (p_s y^{f(h)}_s + p_s K_s C_j \varphi_j)\right)^+ - \sum_{j\in J} \left(p_s A^j_s - (\overline{y}^j_s + p_s K_s C_j)\right)^+ \theta_j \right]$$

which depends both on the default incurred and the default suffered at each state s.

If we denote by $g_s(\theta, \varphi, y_s)$ the insurance net premium of state s given by expression:

$$\sum_{j\in J} \left(p_s A^j_s \varphi_j - (p_s y^{f(h)}_s + p_s K_s C_j \varphi_j)\right)^+ - \sum_{j\in J} \left(p_s A^j_s - (\overline{y}^j_s + p_s K_s C_j)\right)^+ \theta_j,$$

then the overall insurance net premium is $\sum_{s\in S} \beta_s g_s(\theta, \varphi, y_s)$ with $\beta_s > 0 \forall s$. Which will be endogenously determined.

To be more precise, we can think of $\sum_{s\in S} \beta_s g_s(\theta, \varphi, y_s)$ as being the present value of the expected net default, where the expectation is taken with respect the probability vector $\rho = (\frac{\beta_1}{k}, \ldots, \frac{\beta_S}{k})$. This vector is generated by the vector of (actuarially fair) state prices $\beta = (\beta_1, \ldots, \beta_S)$. Here, $k = \sum_{s\in S} \beta_s$ and is interpreted as the discount factor.

2.4 Budget Feasibility

Given the price system $(\pi, p) \in R^{L(S+1)}_+ \times R^J_+$ of assets and commodities, the average output level $\overline{y} \in R^{SJ}_+$ and the state prices $\beta \in R^S_+$; each agent $h \in H$ chooses $(x^h, \theta^h, \varphi^h, y^{f(h)})$, where x^h is the quantity of consumption for agent h, subject to the following budget constraints:

$$p_o(x_o - y_o) + \pi\theta + \sum_{s\in S} \beta_s g_s(\theta, \varphi, y_s) \leq p_o \omega^h_o + \pi\varphi, \tag{1}$$

$$p_s x^h_s + D_s(\varphi, y_s) \leq p_s \omega^h_s + R_s \theta + p_s y^{f(h)}_s + p_s K_s (x_o - y_o), s \in S \tag{2}$$

[6] A potential extension of this framework is the introduction of intermediaries who specialise in insuring lenders.

Budget constraint (1) implies that agent h uses its resources to purchase consumption goods and inputs for contingent future production plus collateralised assets and default insurance contracts. All this expenditure is funded by the value of the initial endowment plus the sales of collateralised assets. Budget constraint (2) states that, in each state of nature s in the second period, agent h cannot spend more on the purchase of commodities x_s and asset deliveries $D_s(\varphi, y_s)$ than the revenue they get from commodity sales (including initial endowments ω_s^h and depreciated collateral $p_s K_s(x_o - y_o)$) and the asset receipts $R_s\theta$.

In addition, the following collateral constraint is satisfied:

$$C\varphi := \sum_{j\in J} C_j \varphi_j \leq x_o - y_o. \tag{3}$$

Hence, we define the budget set for each agent h to be the set

$$B^h(p, \pi, \overline{y}, \beta) = \{(x, \theta, \varphi, y) : (1), (2) \text{ and } (3) \text{ are satisfied}\}$$

2.4.1 Default insurance and asset prices

We now look at the effect that the insurance net premium has on asset prices. To do that, it is convenient to interpret the first-period budget constraint in terms of the default-insurance using the discount factor k defined above and the probability vector ρ.

We write $\sum_{s\in S} \beta_s g_s(\theta, \varphi, y_s)$ as the difference between two terms. The first is the expected (evaluated at probabilities ρ and denoted E_ρ) present value of future default incurred by agents (i.e. those that issue the assets and hence *short*):

$$\pi_{short}(\varphi, y) = kE_\rho[\sum_{j\in J} \left(pA^j\varphi_j - (py^{f(h)} + pKC_j\varphi_j)\right)^+]].$$

The second term is the expected present value of future losses due to default suffered (by those who buy the assets and hence *long*):

$$\pi_{long}\theta = kE_\rho[\sum_{j\in J} \left(pA^j - (\overline{y}^j + pKC_j)\right)^+ \theta_j].$$

So, (1) we can be re-written as:

$$p_o(x_o - y_o) + (\pi - \pi_{long})\theta \leq p_o\omega_o^h + [\pi\varphi - \pi_{long}(\varphi, y)]. \tag{1'}$$

Expression (1') means that the investors who purchase assets receive a discount on the price of every unit of asset purchased. The issuers of the security, on the other hand, get a reduction in their total borrowing. This reduction $\pi_{long}(\varphi, y)$ depends on the level of production they decide and also on the portfolio φ sold. Hence, the higher the level of production, the higher the amount of borrowing thus encouraging borrowers to increase their production.

Finally, we can summarise our production economy with exogenous collateral and default by the following array:

$$\mathcal{E} = [(U^h, \omega^h, Y^{f(h)})_{h\in H}, (A, C, K)]$$

which consists of household-producers, endowments, and collateralised assets which depreciate according to function K. Note that if $Y^{f(h)} = \{0\}$ then an agent h owning firm f is a simple consumer.

2.5 Collateral Equilibrium with Production

Let $R^m = (R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)})^H$ and $R^n := R_+^{L(S+1)} \times R_+^J \times R_+^{SJ}$.

Definition 1. *The allocation $(x^h, \theta^h, \varphi^h, y^{f(h)})_{h\in H} \in R^m$ together with a price system and an average output level $(p, \pi, \overline{y}) \in R_+^n$ is a production-collateral equilibrium for economy $\mathcal{E}$ if the following conditions are satisfied:*

1. *Choices are optimal. That is, for each $h \in H$,*

$$(x^h, \theta^h, \varphi^h, y^{f(h)}) \textit{ maximise } U^h(x^h) \textit{ subject to } B^h(p, \pi, \overline{y}).$$

2. *Market clearing conditions hold:*

$$\sum_H \left(x_0^h - \omega_0^h - y_0^{f(h)}\right) = 0,$$

$$\sum_H \left(x_s^h - K_s(\omega_o^h) - y_s^{f(h)} - \omega_s^h\right) = 0, s \in S,$$

$$\sum_{h\in J} \theta^h = \sum_{h\in J} \varphi^h.$$

3. *For each $s \in S$ and for each j,*

$$\overline{y}_s^j \sum_{h\in H} \theta_j^h = \sum_{h\in H} p_s y_s^{f(h)}.$$

4. $\forall j, s$, *one has that,*

$$\sum_{h\in\mathcal{S}_s^j} \left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j)\right)^+ \theta_j^h = \sum_{h\in\mathcal{D}_s^j} \left(p_s A_s \varphi_j^h - (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j^h)\right)^+,$$

where $\mathcal{D}_s^j = \{h \in H : p_s A_s \varphi_j^h > (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j^h)\}$ *is the set of agents that incur default in state of nature* s *on asset* j*, and* $\mathcal{S}_s^j = \{h \in H : p_s A_s > (\overline{y}_s^j + p_s K_s C_j)\}$ *is the set of agents that suffer default in state of nature* s *on asset* j*.*

Remark 2: Items 1 and 2 are the usual conditions in the classical definition of equilibrium. Item 3 states that, in equilibrium, the average output level $\overline{y}_s^j$, treated as given by individuals, is perfectly anticipated as a proportion of the aggregate output of the economy. Finally, Item 4 states that the default incurred by asset issuers must be equal to the default suffered by lenders.

2.6 Two useful technical results

The following two results are just technical lemmas that will be important later on when we prove that, along the truncated economy, markets in the second period clear. Here we simply state these lemmas and defer their proofs to the appendix.

Lemma 1. *If asset markets clear, then Item 4 of definition 1 implies that:*

$$\sum_{h\in H} R_s^j \theta_j^h = \sum_{h\in H} D_s^j(\varphi_j^h, y_s^{f(h)}), \forall j, s.$$

That is to say, all receipts from asset purchases must equal all deliveries from assets sold.

Lemma 2. *Item 3 of Definition 1 together with the absence of excess demand in asset and insurance markets (i.e.,* $\sum_{h\in H}(\theta^h - \varphi^h) \leq 0$ *and* $\sum_{h\in H} g_s(\theta^h, \varphi^h, y_s^{f(h)}) \leq 0$*), imply that the default suffered must be equal to default incurred in state of nature* s *on asset* $j \in J$*. That is, Item 4 of Definition 1 is true:*

$$\sum_{h\in\mathcal{S}_s^j} \left(p_s A_s^j - (\overline{y}_s^j + p_s K_s C_j)\right)^+ \theta_j^h = \sum_{h\in\mathcal{D}_s^j} \left(p_s A_s^j \varphi_j^h - (p_s y_s^{f(h)} + p_s K_s C_j \varphi_j^h)\right)^+$$

The first result states that, if default incurred and suffered are equal, all deliveries for each portfolio sold must equal total receipts for each portfolio purchased whenever the asset market clearing condition holds. Conversely, the equality between default incurred and default suffered in state of nature s on asset $j \in J$ is always obtained the absence of excess demand in asset and insurance markets (i.e., $\sum_{h\in H}(\theta^h - \varphi^h) \leq 0$ and $\sum_{h\in H} g_s(\theta^h, \varphi^h, y_s^{f(h)}) \leq 0$).

3 Equilibrium Existence

We now tackle the issue of equilibrium existence and non-triviality. For this, we add four extra assumptions. The first three of these are standard in economies with production (see, for instance, Magill and Quinzii, 1996 and Drèze, Minelli and Tirelli, 2008):

H1 The initial endowments of each individual $h \in H$ are strictly positive: $\omega^h \in R_{++}^{L(S+1)}$;

H2 $\forall h \in H$, $u^h : R_+^L \times R_+^{LS} \to R$ is concave, continuous and strictly increasing;

H3 $\forall v \in R_+^{S+1}$, where v is any vector of non-negative constants, the set $(v + \sum_{h \in H} Y^{f(h)}) \cap R_+^{L(S+1)}$ is compact;

H4 $C_j \in R_+^L \setminus \{0\}$.

H4 simply states that collateral must be non zero as is normally assumed in classic collateral models without production (see Geanakoplos and Zame, 2014).

3.1 Non-Triviality of Equilibrium

It is well known that collateral requirements partially protect lenders because receipts are always non-zero due to H4 above. However, this does not preclude the possibility that lenders have over-pessimistic beliefs about receipts because the collateral could experience a large depreciation in the second period.

In our model, we can find a trivial equilibrium in the following way: suppose that the prices of assets are $\pi = 0$, the average output level $\overline{y} = 0$ and the collateral requirements are small enough so that the receipts $R_s^j = \min\{p_s A_s^j, \overline{y}_s^j + p_s K_s C_j\} \to 0$, implying $\theta_j^h = 0, \forall h$. Thus, $\varphi_s^h = 0, \forall h$ which implies that $D_s^h(\varphi_j^h, y_s^{f(h)}) = 0, \forall s, j$ since $y_s^{f(h)} = 0, \forall h$ once we have fixed $\overline{y}_s^j$ to be zero for every j, s. This way our model reduces to the model with durable goods and without trading in the financial markets whose existence is proven in Geanakoplos and Zame (2014).

If there is trading in the asset markets in equilibrium (i.e., $\sum_{h \in H} \theta_j^h = \sum_{h \in H} \varphi_j^h \neq 0$), then lenders will rationally anticipate the average level of production as $\overline{y}_s^j = \frac{\sum_{h \in H} p_s y_s^{f(h)}}{\sum_{h \in H} \varphi_j^h}$. However, when assets are not traded, the expected average level of production $\overline{y}_s^j$ cannot be determined from the earlier formula. This issue was firstly addressed in GEI models with default penalties and without production by Dubey et al. (1990) and Araujo et al. (1998) and in an exogenous collateral setting without production by Steinert and Torres-Martínez (2004).

We next adapt the concept of non-triviality of equilibria to our context following Araujo et al. (1998). We first define a no-trade equilibrium as the one in which $\theta^h = 0 = \varphi^h, \forall h$. Secondly, we define the payment rate t^j_s of asset j in state s as being $\min\{1, \frac{p_s K_s C_j + \overline{y}^j_s}{p_s A^j_s}\}$. Hence, R^j_s can be written as $R^j_s = t^j_s p_s A^j_s, \forall j, s$.

Notice that t^j_s could be equal to 1 independently of the endogenous expected output level $\overline{y}^j_s$. This occurs when the future value of the depreciated collateral is large enough so that $p_s K_s C_j \geq p_s A^j_s$. Otherwise, the endogenous expected output level $\overline{y}^j_s$ must be large enough in order to make $p_s K_s C_j + \overline{y}^j_s > p_s A^j_s$. Consequently, when the collateral is sufficiently valued it eliminates any investors' pessimism with respect to the payment rate. Otherwise, the expectation about the payment rate of the economy depends on its aggregate level of output (GDP) so that a high aggregate output would also eliminate any investors' pessimism.

Following Araujo et al. (1998), thus, we can state the following definition:

Definition 2. *An equilibrium is said to be non-trivial if there is trading in the asset markets or, when assets are not traded, the payment rates* $t^j_s = 1, \forall s, j$.

We are now ready to state our existence result:

Theorem 1. *Under assumptions H1-H4, for every collateral-production economy* $\mathcal{E}$ *a non-trivial production-collateral equilibrium exists.*

The proof of our result follows the general lines of the arguments in Araujo et al (2000, 2005). See appendix for full details.

4 Financial Policies

The purpose of this section is firstly to determine whether the financial policies of the firm have an impact on the real variables of the economy. That is, whether or not they are relevant. Secondly, we also study whether, as it is standard in many models in macroeconomics, it is possible to separate the roles of agents as consumers and producers who only decide their production and funding policies. The former is known as the Modigliani-Miller Theorem and the latter as the Fisher Separation Theorem.

To do so, we specialise our analysis by considering the finance economy associated to the multi-goods economy of the previous section. In this new version, we will distinguish two kinds of equilibria. On the one hand, a reduced-form equilibrium where consumption and production activities are not separated. In other words, the roles of an agent as consumer and entrepreneur are undistinguishable. On the other hand, an extensive-form equilibrium where, in addition of separating consumption and production roles, the financial accounts of consumers and firms are also separated.

4.1 The Finance Economy

Following Magill and Quinzii (1996), the finance economy is obtained by fixing the equilibrium commodity prices $p \in R_+^{L(S+1)}$ of the corresponding multi-goods economy. Consequently, any plan, be it of consumption or production, involving the L goods is replaced by an income stream. Thus, the commodity space is now R^{1+S}; the utility function of each agent h is $U^h : R_+^{1+S} \to R$; and the technology describing all the feasible investment projects is $Y^{f(h)} \subset R^{1+S}$. All income streams will be denoted by lowercase letters, which are also used to denote the values of collateral, $c = (c_o, c_1, ..., c_S)$ and the value of promises that are now represented by $r = (r_1, r_2, ..., r_S)$ where each $r_s \in R_+^J$ stands for the payoff of the portfolio containing J assets. Finally, k_s stands for the depreciation in value of goods used for consumption, production, or collateral.

Each element y belonging to $Y^{f(h)}$ describes the income streams $y = (y_o, y_1, \ldots, y_S)$ generated by investment projects available to firm $f(h)$. We assume that the set of investment projects $Y^{f(h)}$ is represented by a transformation $T : R^{1+S} \to R$. Since, by definition, $Y^{f(h)} = \{y \in R^{(S+1)} : T(y) \leq 0\}$ then $y \in Y^{f(h)}$ if and only if $T(y) \leq 0$. The transformation T is assumed to be non-decreasing, quasi-convex, and differentiable. The utility function is also assumed to be differentiable and to have a degree of concavity so that the Kuhn-Tucker conditions to solve an optimisation problem associated to the consumer or the firm are necessary and sufficient.

We can write agent h's problem as follows:

$$\max_{(x,y,\theta,\varphi)} U^h(x_o, x_s)$$

s.t.

$$x_o - y_o - \omega_o^h + \pi(\theta - \varphi) + \sum_{s\in S} \beta_s \sum_{j\in J} g_s^j(\varphi_j, \theta_j, y_s) \leq 0, \tag{2'}$$

$$x_s - y_s - \omega_s^h \leq \sum_{j\in J} R_s^j \theta_j - \sum_{j\in J} D_s^j(\varphi_j, y_s) + k_s(x_o - y_o), s \in S, \tag{3'}$$

$$c_o\varphi \leq x_o - y_o \tag{4'}$$

$$T(y_o, y_s) \leq 0 \tag{5'}$$

where R^j_s, $D^j_s(\varphi_j, y_s)$ and $g^j_s(\theta, \varphi, y_s)$ are defined, respectively, as

$$R^j_s = \min\{r^j_s, \overline{y}^j_s + c_{sj}\},$$

$$D^j_s(\varphi_j, y_s) = \min\{r^j_s \varphi_j, y_s + c_{sj}\varphi_j\},$$

$$g_s(\theta, \varphi, y_s) = \sum_{j \in J} g^j_s(\theta_j, \varphi_j, y_s),$$

with

$$g^j_s(\varphi_j, \theta_j, y_s) = \left(r^j_s \varphi_j - (y^{f(h)}_s + c_{sj}\varphi_j)\right)^+ - \left(r^j_s - (\overline{y}^j_s + c_{sj})\right)^+ \theta_j.$$

Thus, we can re-write g^j_s as follows:

$$g^j_s(\varphi_j, \theta_j, y_s) = r^j_s \varphi_j - D^j_s(\varphi_j, y_s) - (r^j_s - R^j_s)\theta_j$$

In the rest of the paper, we will use the symbol $\partial_x f$ to denote the partial derivative of function f with respect to variable x if the function is differentiable. If f is convex or concave, then the symbol will denote the super-gradient or sub-gradient when f is not differentiable. More precisely, $\partial_x D^j_s$ is the sub-gradient with respect to x, since the function $\min\{.,.\}$ is convex.

4.2 Reduced-form equilibrium

The following notion of equilibrium is practically the same as the one in Definition 1, but now it is given in nominal terms and the commodity market clearing conditions are implied by the asset market clearing conditions. More precisely, we have:

Definition 3. *An allocation* $\left((x^h, \theta^h, \varphi^h); (y^{f(h)})\right)_{h \in H} \in R^{L(S+1)}_+ \times R^J_+ \times R^J_+ \times R^{L(S+1)}_+$ *together with* $(\pi, \beta, \overline{y})$ *is said to be a reduced-form equilibrium with production and default if the following conditions are satisfied:*

1. *Asset Markets clear:*
   $\sum_H \theta^h = \sum_{h \in H} \varphi^h$.

2. *For each* $h \in H$, $\left((x^h, \theta^h, \varphi^h); y^{f(h)}\right)$ *maximizes* $U^h(x_o, x_{-o})$ *subject to budget constraints (2'), (3') and (4').*

3. *For every* j, s, *one has:*

$$\overline{y}^j_s \sum_{h \in H} \theta^h_j = \sum_{h \in H} y^{f(h)}_s$$

4. $\forall j, s$, *one has that:*

$$\sum_{h \in \mathcal{S}_s^j} \left(r_s^j - (\overline{y}_s^j + c_{sj}) \right)^+ \theta_j^h = \sum_{h \in \mathcal{D}_s^j} \left(r_s^j \varphi_j^h - (y_s^{f(h)} + c_{sj} \varphi_j^h) \right)^+,$$

where $\mathcal{D}_s^j = \{h \in H : r_s^j \varphi_j^h > (y_s^{f(h)} + c_{sj}\varphi_j^h)\}$ *is the set of agents that incur default in state of nature* s *on asset* j, *and* $\mathcal{S}_s^j = \{h \in H : r_s^j > (\overline{y}_s^j + c_{sj})\}$ *is the set of agents that suffer default in state of nature* s *on asset* j.

It is useful to note that, since in a reduced-form equilibrium default and bankruptcy cannot be distinguished, the financial roles of agents cannot be distinguished either. For this reason, we will keep using the term "default" for the reduced-form equilibrium since it is the individuals, as borrowers, who do not honour their financial commitments. Later, we will be able to separate the non-payment decisions of consumers and firms. When we do this, we will use the term "default" for consumers and "bankruptcy" for firms (entrepreneurs). However, individuals as investors (lenders) will only suffer default regardless of who is not honoring their financial commitments.

4.3 Modified reduced-form equilibrium

Since our objective is to compare the different types of equilibria defined earlier, we will assume that φ can be decomposed in two terms: $\varphi = \xi + \phi$, where ξ and ϕ are the amounts of borrowing of agents as consumers and firms respectively. Given this, and changing x_o for $x_o + c_o\xi$ and y_o for $y_o - c_o\phi$, we can write agent h's problem as:

$$\max_{(x,y,\theta,\xi,\phi)} U^h(x_o + c_o\xi, x_{-o})$$

s.t.

$$x_o + c_o\xi - (y_o - c_o\phi) - \omega_o^h + \pi(\theta - \xi) - \pi\phi + \sum_{s \in S} \beta_s \sum_{j \in J} g_s^j(\xi + \phi, \theta, y_s) \leq 0, \qquad (1'')$$

$$x_s - y_s - \omega_s^h \leq \sum_{j \in J} R_s^j \theta_j - \sum_{j \in J} D_s^j(\xi_j + \phi_j, y_s) + k_s(x_o - y_o) + c_s(\xi + \phi), s \in S, \qquad (2'')$$

$$T(y_o - c_o\phi, y_s) \leq 0 \qquad (3'')$$

where x_o and y_o are now consumption and (legitimate) inputs which are assumed to depreciate completely (i.e., $k_s(x_o - y_o) = 0$). $c_o\xi$ is the collateral used for consumption (e.g. a car) and $c_o\phi$ is the collateral used in the production process (e.g. machinery). Finally, the value of collateral in the second period is defined as $c_s \in R_{++}^J$.

Definition 4. *An allocation* $\left((x^h,\theta^h,\xi^h);(y^{f(h)},\phi^{f(h)})\right)_{h\in H} \in R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)}$ *together with* $(\pi,\beta,\overline{y})$ *is said to be a reduced-form equilibrium with production and default*[7] *if the following conditions are satisfied:*

1. *Asset Markets clear:*

   $\sum_H \theta^h = \sum_{h\in H} \xi^h + \phi^{f(h)}$.

2. *For each* $h \in H$, $\left((x^h,\theta^h,\xi^h);(y^{f(h)},\phi^{f(h)})\right)$ *maximizes* $U^h(x_o + c_o\xi, x_{-o})$ *subject to budget constraints (1"), (2") and (3").*

3. *For every* j, s, *one has:*
$$\overline{y}^j_s \sum_{h\in H} \theta^h_j = \sum_{h\in H} y^{f(h)}_s$$

4. *Item 4 of Definition 3 holds with* $\varphi^h = \xi^h + \phi^{f(h)}$. *That is:*
$$\sum_{h\in\mathcal{S}^j_s} \left(r^j_s - (\overline{y}^j_s + c_{sj})\right)^+ \theta^h_j = \sum_{h\in\mathcal{D}^j_s} \left(r^j_s(\xi^h + \phi^{f(h)}) - (y^{f(h)}_s + c_{sj}(\xi^h + \phi^{f(h)}))\right)^+.$$

The right hand side of Item 4 above can be written as:
$$\sum_{h\in\mathcal{D}^j_s} [r^j_s(\xi^h + \phi^{f(h)}) - D^j_s(\xi^h + \phi^{f(h)}, y_s)].$$

We write the Lagrangian associated to the individual's problem defined above as:
$$L = U^h(x_o + c_o\xi, x_{-o}) - \alpha^h_o[x_o + c_o\xi - (y_o - c_o\phi) - \omega^h_o + \pi(\theta - \xi)$$
$$-\pi\phi + \sum_{s\in S}\beta_s \sum_{j\in J} g^j_s(\xi_j + \phi_j, \theta_j, y_s)] - \sum_{s\in S}\alpha^h_s[x_s - y_s - \omega^h_s - c_s(\xi + \phi)$$
$$+\sum_{j\in J} D^j_s(\xi_j + \phi_j, y_s) - \sum_{j\in J} R^j_s\theta_j] - \nu T(y_o - c_o\phi, y_s).$$
where α_o is the Lagrange multiplier of the first period budget constraint (1"), α_s are the Lagrange multipliers of the state-contingent second period budget constraint (2"), and ν is the Lagrange multiplier of the technological constraint (3").

The first order conditions for this problem are:

1. x_o :
$$\partial_{x_o} U^h(x^h_o + c_o\xi^h, x^h_{-o}) = \alpha^h_o \tag{4}$$

[7] Note that we mean "default" because it is agent h who does not honour financial commitments.

2. x_s :

$$\partial_{x_s} U^h(x_o^h + c_o\xi^h, x_{-o}^h) = \alpha_s^h \quad (5)$$

3. ξ_j :

$$\partial_o U^h(x_o^h + c_o\xi^h, x_{-o}^h)c_o - \alpha_o^h(c_{oj} - \pi_j) + \sum_{s\in S}\partial_{\xi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) +$$

$$+\sum_{s\in S}\alpha_s^h[c_{sj} - \partial_{\xi_j} D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s)] = 0, \quad (6)$$

4. θ_j :

$$\alpha_o^h\pi_j + \sum_{s\in S}\beta_s\partial_{\theta_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = \sum_{s\in S}\alpha_s^h R_s^j, s \in S. \quad (7)$$

5. y_o :

$$\alpha_o = \nu\partial_{y_o} T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \quad (8)$$

6. y_s :

$$\alpha_s^h\left(1 - \sum_{j\in J}\partial_{y_s} D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s^{f(h)})\right) - \alpha_o^h\beta_s\sum_{j\in J}\partial_{y_s} g_s^j(\xi_j + \phi_j, \theta_j, y_s)$$

$$= \nu\partial_s T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)}), s \in S. \quad (9)$$

7. ϕ_j :

$$\alpha_o[\pi_j - c_{oj} - \sum_{s\in S}\beta_s\partial_{\phi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s)] +$$

$$\sum_{s\in S}\alpha_s^h[c_{sj} - \partial_{\phi_j} D_s^j(\xi_j^h + \phi_j^{f(h)}, y_s)] + \nu\partial_o T(y_o^{f(h)} - c_o\phi^{f(h)}, y_{-o}^{f(h)})c_{oj} = 0. \quad (10)$$

The following lemma concerns the financial decisions of consumers and firms as borrowers and therefore their deliveries. Now both "default" and "bankruptcy" are the decision of borrowers. However, investors will suffer default whenever either consumers default or firms go bankrupt. The proof of the following lemma is straightforward so it will be excluded.

Lemma 3. *(Separation of Deliveries)*

1. *If there is bankruptcy for asset j and state s, i.e. $y_s + c_{sj}\phi_j \leq r_s^j\phi_j$, then there is default for asset j and state s. Moreover, for any $\xi_j > 0$*

$$D_s^j(\xi_j + \phi_j, y_s) = d_s^j\xi_j + D_s^j(\phi_j, y_s) \quad (I),$$

where $d_s^j = \min\{r_s^j, c_{sj}\}$.

2. *If there is no bankruptcy, then the previous equality holds provided that there is no default. That is,* $r_s^j \leq c_{sj}$.

Remark 3: Notice that there could be default (i.e., $c_{sj} < r_s^j$) without having bankruptcy. This is true since the firm could produce a y_s large enough such that $y_s + c_{sj}\phi_j > r_s^j\phi_j$ so that (I) no longer holds. An immediate consequence of (I) is that the insurance net premium is given by

$$g_s^j(\xi_j + \phi_j, \theta_j, y_s) = (r_s^j - d_s^j)\xi_j - (r_s^j - R_s^j)\theta_j + \left(r_s^j\phi_j - D_s^j(\phi_j, y_s)\right). \qquad (II)$$

The following corollary is thus straightforward:

Corollary 1. *If the separation of deliveries is true, then the following relations hold:*

1. $\partial_x D_s^j(\xi_j + \phi_j, y_s) = \partial_x D_s^j(\phi_j, y_s), x \in \{\phi_j, y_s\}$,
2. $\partial_{\xi_j} D_s^j(\xi_j + \phi_j, y_s) = d_s^j$,
3. $\partial_{\phi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = r_s^j - \partial_{\phi_j} D_s^j(\phi_j, y_s)$,
4. $\partial_{y_s} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = -\partial_{y_s} D_s^j(\phi_j, y_s)$,
5. $\partial_{\xi_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = r_s^j - d_s^j$,
6. $\partial_{\theta_j} g_s^j(\xi_j + \phi_j, \theta_j, y_s) = -(r_s^j - R_s^j)$.

4.4 Extensive-form equilibrium

When there is no default or bankruptcy, we can define an equilibrium where consumption and production activities are separated. This kind of equilibrium is called "profit-maximising equilibrium" where firms maximize their profits (see Magill and Quinzii, 1996). However, in our setting, due to the presence of collateral and the possibility of bankruptcy, firms, in general, do not maximize profits unless some conditions are satisfied which we will discuss later on.

4.4.1 Separation of Financial Policies and the Firm's problem

Suppose that we can decompose delivery decisions $D_s(\xi + \phi, y)$ as in Lemma 3. Then, given the present-value vector $\beta^h = (\frac{\alpha_o^h}{\alpha_o^h}, \ldots, \frac{\alpha_S^h}{\alpha_o^h}) \in R_{++}^{1+S}$, we can formulate the following problem for the firm to solve:

$$\max_{y,\phi} \pi\phi + (y_o - c_o\phi) - \sum_{s \in S} \beta_s \sum_{j \in J} \left(r_s^j\phi_j - D_s^j(\phi_j, y_s)\right) + \sum_{s \in S} \beta_s^h \left(y_s + c_s\phi - \sum_{j \in J} D_s^j(\phi_j, y_s)\right)$$

s.t .

$$T(y_o - c_o\phi, y_s) \leq 0.$$

That is, the firm $f(h)$ maximizes the present value of all its cash flows discounted by the the present-value vector β^h of its owner h.

Note that the term $\sum_{s\in S}\beta_s\sum_{j\in J}(r_s^j\phi_j - D_s^j(\phi_j, y_s))$ is the part of the insurance net premium, given by (II), paid by the firm.

If we denote by μ the Lagrange multiplier associated to the technological constraint above, the first order conditions for firm's problem are:

1. y_o :

$$1 = \mu\partial_o T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \tag{11}$$

2. y_s :

$$\beta_s^h + (\beta_s - \beta_s^h)\sum_{j\in J}\partial_{y_s}D_s^j(\phi_j, y_s) = \mu\partial_s T(y_o^{f(h)} - c_o\phi^{f(h)}, y_{-o}^{f(h)}) \tag{12}$$

3. ϕ_j :

$$\pi_j - c_{oj} - \sum_{s\in S}\left(\beta_s r_s^j - \beta_s^h c_{sj}\right) + \sum_{s\in S}(\beta_s - \beta_s^h)\partial_{\phi_j}D_s^j(\phi_j, y_s)$$
$$+\mu\partial_o T(y_o^f(h) - c_o\phi^{f(h)}, y_{-o}^{f(h)})c_{oj} = 0 \tag{13}$$

Remark 4: Using Corollary 1 we can verify that (8), (9) and (10) together are equivalent to (11), (12) and (13) also together. This motivates the following notion of equilibrium where consumption and production activities are separated.

Definition 5. *An allocation* $[(x^h, \theta^h, \xi^h); (y^{f(h)}, \phi^{f(h)})]_{h\in H} \in R_+^{L(S+1)} \times R_+^J \times R_+^J \times R_+^{L(S+1)} \times R_+^J$ *together with* $(\pi, \beta, \overline{y})$ *is said to be an extensive-form equilibrium with default and bankruptcy if the following conditions are satisfied:*

1. *Asset Markets clear:*

   $\sum_H \theta^h = \sum_{h\in H}\left(\xi^h + \phi^{f(h)}\right).$

2. *For each* $h \in H, (x^h, \theta^h, \xi^h)$ *maximises* U^h *subject to budget constraints (1"), (2") with* $(y^{f(h)}, \phi^{f(h)})$ *taken as given;*

3. *The vector* $(y^{f(h)}, \phi^{f(h)})$ *maximises the firm's problem given in Subsection 4.4.1*

4. *Items 3 and 4 of Definition 4 are satisfied under separation of deliveries given by (I) and separation of the payment of the default-insurance premium given by (II).*

Remark 5: Let $\triangle^{f(h)} \in R^{1+S}$ be the dividend policy of firm $f(h)$. If $\triangle^{f(h)}$ is defined as the pair $\triangle_o^{f(h)} = y_o^{f(h)} + (\pi - c_o)\phi^{f(h)} - \sum_{s\in S}\beta_s \sum_{j\in J}[r_s^j\phi_j^{f(h)} - (y_s^{f(h)} + c_{sj}\phi_j^{f(h)})]^+$ and $\triangle_s^{f(h)} = y_s + c_s\phi^{f(h)} - \sum_{j\in J}\min\{r_s^j\phi^{f(h)}, y_s + c_{sj}\phi_j^{f(h)}\}$, $s \in S$, then we say that portfolio-dividend policy $(\triangle^{f(h)}, \phi^{f(h)})$ balances the investment project $y^{f(h)}$. If we interpret $\triangle^{f(h)}$ as the stream of payments made by the firm $f(h)$ to its owner h, then we can think of Item 2 of Definition 4 as h maximising U^h subject to (1") and (2") where the new income is now $\tilde{w}_s^h = \omega_s^h + \triangle_s^{f(h)}, s = 0, 1, \ldots, S$. Similarly, Item 3 we be viewed as $f(h)$ maximising the present value of its dividends $\sum_{s=0}^S \beta_s^h \triangle_s^{f(h)}$ subject to all portfolio-dividend policies balancing all the feasible invest projects. That is, in this kind of equilibrium, the financial accounts of both firms and their owners ares separated. See Magill and Quinzii (1996) for a ampler discussion on this kind of equilibrium in a setting without default or bankruptcy.

Consequently, if we assume separation in deliveries, one obtains the following theorem:

Theorem 2. *Under separation of deliveries as in Lemma 3, an extensive-form equilibrium with default and bankruptcy is equivalent to a reduced-form equilibrium with default.*

Proof. The proof of this theorem follows from Remark 2 and the equivalence between the first order conditions associated to h's problem defined in Item 2 of Definition 4 and (4), (5), (6) and (7) together. □

4.5 Fisher and Modigliani-Miller

We now discuss the two fundamental results in corporate finance: Fisher's Separation Theorem and Modigliani-Miller's Theorem. In order to get the Fisher Theorem, we need to prove that the reduced-form equilibrium is equivalent to the extensive-form equilibrium where the objective of the firm is the maximization of profits. To obtain this, it is necessary that, for every asset j, $r_s\phi_j = D_s^j(\phi_j, y_s)$. That is to say, that there is no bankruptcy. Thus, combining (11) and (13) the firm's problem of Definition 4 would reduce to:

$$\max_{y,\phi}(y_o - c_o\phi) + \sum_{s\in S}\beta_s^h y_s$$

s.t .

$$T(y_o - c_o\phi, y_s) \leq 0.$$

which is the legitimate profit maximization problem.

Otherwise, if there is strict bankruptcy (i.e., $\exists j : r_s^j \phi_j > y_s + c_{sj}\phi_j$), then Fisher's separation theorem cannot hold anymore unless the firm goes bankrupt on only one asset j and $\beta_s = \beta_s^h$. The argument is very similar to the earlier one yielding the same problem as above.

Finally, to obtain the Modigliani-Miller Theorem, which relates to the indeterminacy of the financial policies of firm, we need that ϕ does not affect either the present value of dividends of the firm or the budget set of the entrepreneur. Clearly, this will not be true unless there is no bankruptcy or default (see also Magill and Quinzii, 1996).

In our model, the equivalence between the modified reduced-form equilibrium and extensive-form equilibrium given by Theorem 2 is the simplest version of the well-known Modigliani-Miller Theorem which states that the precise nature of a firm's financial policy does not matter provided that it finances the firm's productive activities. The result of Theorem 2 (comparison of equilibria), however, depends strongly on the separation of deliveries which fails, for instance, when there is default without bankruptcy. This situation could happen, for instance, when the collateral has devalued below the payoff but the level of production is sufficiently high so that $c_s + y_s > r_s$. In this case, there would be no bankruptcy and the separation of deliveries would not be satisfied, making Theorem 2 invalid. Therefore, the financial policies of firms would, in general, be relevant.

5 Conclusions

We present an incomplete markets general equilibrium model of default and bankruptcy in an setting involving exogenous collateral and, importantly, endogenous production. The collateral applies to the whole of the commodities available for consumption and production in the economy. Because the owners of producing firms have unlimited liability, in case of default, lenders can seize the value of collateral and future production. This makes deliveries and default/bankruptcy decisions endogenous. Because of this endogeneity, we make receipts anonymous in the sense that lenders have a portfolio of assets issued by producers but cannot anticipate the identity of the firm(s) defaulting. Thus, lenders anticipate a default rate that depends on the average level of production (default is counter-cyclical). That is, we assume a type of tranching because lenders, in case of default, only expect to receive a fraction of the output of the firm and the value of collateral. Finally, in order to avoid a zero production equilibrium, we introduce an insurance mechanism.

We show that, under usual assumptions on utilities and technologies our economy has a non-trivial equilibrium. Hence, our contribution is to establish a theoretical framework stressing the role of the production sector, which is absent in earlier studies with default and collateral, and to show that default and bankruptcy are consistent with the orderly functioning of markets. In analysing the non-triviality of equilibrium, we show that a high aggregate level of economic activity could eliminate any investors' pessimism.

Finally, we study the financial policies of the firm. We conclude that the two main theorems in corporate finance, namely Fisher's Separation and Modigliani-Miller, are not satisfied unless we impose very restrictive assumptions on the trading of portfolios and on the way borrowers repay their debts. Thus, in general, the separation between the financial decisions of firms and their owners is not possible, and the financial policies of firms will be relevant for real outcomes in the economy.

Appendix

Proof of Lemma 1

Proof. Firstly, we will prove that $\sum_{h\in H} R^j_s\theta^h_j \leq \sum_{h\in H} D_s(\varphi^h_j, y^{f(h)}_s)$. The opposite case is proven similarly. By contradiction, suppose that

$$\sum_{h\in H} R^j_s\theta^h_j > \sum_{h\in H} D_s(\varphi^h_j, y^{f(h)}_s).$$

From Item 4 of Definition 1, one has that the set $\mathcal{S}^j_s$ is either H or empty. Let us assume the first case, which is more interesting. Rewriting the right-hand side of the previous inequality and using the definition of both $\mathcal{S}^j_j$ of D^j_s, we have

$$\sum_{h\in H}(\overline{y}^j_s + p_s K^j_s C_j \theta^h_j) > \sum_{h\in \mathcal{D}^j_s}(p_s y^{f(h)}_s + p_s K_s C_s \varphi^h_j) + \sum_{h\in (\mathcal{D}^j_s)'} p_s A^j_s \varphi^h_j.$$

After multiplying by a minus sign and after adding and subtracting the term $\sum_{h\in\mathcal{D}^j_s} p_s A^j_s\varphi^h_j$ in the right-hand side, we obtain

$$-\sum_{h\in H}(\overline{y}^j_s + p_s K^j_s C_j \theta^h_j) < -\sum_{h\in \mathcal{D}^j_s}(p_s y^{f(h)}_s p_s K_s C_s \varphi^h_j) - \sum_{h\in (\mathcal{D}^j_s)'} p_s A^j_s \varphi^h_j - \sum_{h\in \mathcal{D}^j_s} p_s A^j_s \varphi^h_j + \sum_{h\in \mathcal{D}^j_s} p_s A^j_s \varphi^h_j$$

Grouping in convenient way and using the asset market clearing condition one has

$$\sum_{h\in H}\left(p_s A^j_s\theta^h_j - (\overline{y}^j_s + p_s A^j_s C_j \theta^h_j)\right) < \sum_{h\in \mathcal{D}^j_s}\left(p_s A^j_s \varphi^h_j - (p_s y^{f(h)}_s + p_s K_s C_s \varphi^h_j)\right),$$

which contradicts Item 4 of definition 1. Then Lemma 1 follows. □

Proof of Lemma 2

Proof. Firstly, we prove that Item 3 of Definition 1 together with the absence of excess demand in asset markets imply that

$$\sum_{h\in\mathcal{S}_s^j}\left(p_sA_s^j-(\overline{y}_s^j+p_sK_sC_j)\right)^+\theta_j^h\leq\sum_{h\in\mathcal{D}_s^j}\left(p_sA_s^j\varphi_j^h-(p_sy_s^{f(h)}+p_sK_sC_j\varphi_j^h)\right)^+$$

We assume as in Lemma 1's proof that $\mathcal{S}_s^j=H$. Then for asset j we have

$$\sum_{h\in H}\left(p_sA_s-(\overline{y}_s^j+p_sK_sC_j)\right)^+\theta_j^h=\sum_{h\in H}p_sA_s\theta_j^h-\sum_{h\in H}\overline{y}_s^j\theta_j^h-\sum_{h\in H}p_sK_sC_j\theta_j^h$$

Item 3 together with the absence of excess demand in asset markets imply that

$$\sum_{h\in H}\left(p_sA_s-(\overline{y}_s^j+p_sK_sC_j)\right)^+\theta_j^h=\sum_{h\in H}p_sA_s\varphi_j^h-\sum_{h\in H}p_sy_s^{f(h)}-\sum_{h\in H}p_sK_sC_j\varphi_j^h$$

$$\leq\sum_{h\in\mathcal{D}_s^j}\left(p_sA_s\varphi_j^h-(p_sy_s^{f(h)}+p_sK_sC_j\varphi_j^h)\right)+\sum_{h\in(\mathcal{D}_s^j)'}\left(p_sA_s\varphi_j^h-(p_sy_s^{f(h)}+p_sK_sC_j\varphi_j^h)\right)$$

$$\leq\sum_{h\in\mathcal{D}_s^j}\left(p_sA_s^j\varphi_j^h-(p_sy_s^{f(h)}+p_sK_sC_j\varphi_j^h)\right)^+$$

The last step follows because the second term of the last inequality is negative.

To prove the converse inequality, we assume, by reduction to the absurd that

$$\sum_{h\in\mathcal{S}_s^j}\left(p_sA_s^j-(\overline{y}_s^j+p_sK_sC_j)\right)^+\theta_j^h<\sum_{h\in\mathcal{D}_s^j}\left(p_sA_s^j\varphi_j^h-(p_sy_s^{f(h)}+p_sK_sC_j\varphi_j^h)\right)^+.$$

Summing in j in both sides of the earlier inequality and by transposing all to on the right-hand side of the same inequality, we obtain an excess of demand in the default insurance markets which is a contradiction. Thus Lemma 2 follows.

□

Proof of Theorem 1

The proof will be carried out in several steps:

- Step 1. Truncated Economy.

  Define for each positive integer $n\in Z_+$, two sets: $T_n=[0,n]^{L(S+1)}\times[0,n]^J\times[0,n]^J$ and $K_n=\left([-n,n]^{L(S+1)}\cap Y^{f(h)}\right)$. The former contains the choices of individuals as consumers $z^h=(x^h,\theta^h,\varphi^h)$, and the latter as entrepreneurs $y^{f(h)}$. Let us now consider a sequence of economies $\{\mathcal{E}^n\}_{n\in Z_+}$, in which the budget sets of each $h \in H$ is:

$$B_n^h(\pi, p, \overline{y}) = \{(z^h, y^{f(h)}) \in T_n \times K_n : \text{ (1), (2) and (3) are satisfied } \}$$

Let us also consider the price system in the simplex:

$$(p_0, \pi, \beta) \in \triangle^{L+J+S-1} \text{ and } p_s \in \triangle^{L-1}, s \in S.$$

Lastly, assume that $\overline{y}_s^j \in [0, n], \forall s, j$.

- Step 2. The generalised game.

  For each $n \in Z_+$, define the generalised game $\mathcal{G}^n$, played by H individuals and $1+S+SJ$ additional players, where $1+S$ are auctioneers and the other SJ players are fictitious agents.

  1. Each agent $h \in H$ maximises U^h in the constrained strategy set $B_n^h(\pi, p)$.
  2. The auctioneer of the first period chooses $(p_o, \pi, \beta) \in \triangle^{L+J+S-1}$, in order to maximise:

     $$p_0 \sum_H \left(x_0^h - y_0^{f(h)} - \omega_0^h\right) + \pi \sum_H (\theta^h - \varphi^h) + \beta \sum_{h\in H} g(\theta^h, \varphi^h, \tilde{y}^{f(h)})$$

     .
  3. The auctioneer of state s, of the second period, chooses $p_s \in \triangle^{L-1}$ in order to maximise:

     $$p_s \sum_H \left(x_s^h - \omega_s^h - K_s(\omega_o^h) - y_s^{f(h)}\right)$$

     .
  4. Each of the remaining JS fictitious agents chooses $\overline{y}_s^j \in [0, n]$ in order to minimise:

     $$\left(\overline{y}_s^j \sum_{h\in H} \theta_j^h - \sum_{h\in H} p_s y_s^{f(h)}\right)^2$$

     .

The objective functions of the agents are continuous and quasi-concave in their strategies. Furthermore, the objective functions of the auctioneers are continuous and linear in their own strategies, and therefore quasi-concave. The correspondence of admissible strategies, for the agents and for the auctioneers, has compact domain and compact, convex, and nonempty values. Such correspondences are upper semi-continuous, because they have compact values and a closed graph. The lower semi-continuity of interior correspondences follows from Hildenbrand (1974, p. 26, fact 4). Because the closure of a lower semi-continuity correspondence is also lower semi-continuous, the continuity of these set functions is guaranteed. We can then apply Kakutani's fixed point theorem to the correspondence of optimal strategies in order to find a pure strategy equilibrium for $\mathcal{G}_n$.

- Step 3. Existence of equilibrium for each truncated economy.

  We shall show that for n large enough a pure strategy equilibrium for $\mathcal{G}_n$ is also a collateral equilibrium for $\mathcal{E}^n$. To avoid cluttering notation, all indexes n of equilibrium variables of $\mathcal{G}_n$ will momentarily be dropped.

  Let $(z^h, y^{f(h)}) \in T_n \times K_n, (p_o, \pi, \beta) \in \triangle^{L+J+S-1}, p_s \in \triangle^{J-1}, \overline{y} \in [0,n]^{SJ}$ be a pure strategy equilibrium for $\mathcal{G}_n$.

  From Item 1 of Step 2 it follows that

$$p_o x_o^h - p_o y_o^{f(h)} + \pi\theta^h + \sum_{s\in S} \beta_s g_s(\theta^h, \varphi^h, \tilde{y}_s^{f(h)}) \le p_o\omega_o^h + \pi\varphi^h,$$

$$p_s x_s^h + D_s(\varphi^h, \tilde{y}^{f(h)}) \le p_s\omega_s^h + R_s(\theta^h, \overline{y}) + p_s y_s^{f(h)} + p_s K_s(x_o^h - y_o^{f(h)}), s \in S,$$

$$C\varphi^h := \sum_{j\in J} C_j\varphi_j^h \le x_o^h - y_o^{f(h)}.$$

  and $U^h(x^h) \ge U^h(x)$, for all budget feasible x.

  On the other hand, the optimality conditions of the auctioneers' problems imply that

$$\sum_H \left(x_0^h - y_0^{f(h)} - \omega_0^h\right) \le 0, \sum_H \left(\theta^h - \varphi^h\right) \le 0, \sum_{h\in H} g(\theta^h, \varphi^h, \tilde{y}^{f(h)}) \le 0 \qquad (14)$$

$$\sum_H \left(x_s^h - \omega_s^h - K_s(\omega_o^h) - y_s^{f(h)}\right) \le 0 \qquad (15)$$

  Moreover,

$$\overline{y}_s^j \sum_{h\in H} \theta_j^h = \sum_{h\in H} p_s y_s^{f(h)}. \qquad (16)$$

Adding in h and grouping in a convenient way the budget constraints above, we have

$$p_o \sum_{h \in H} (x_o^h - y_o^{f(h)} - \omega_o^h) + \pi \sum_{h \in H} (\theta - \varphi^h) + \beta \sum_{h \in H} g(\theta^h, \varphi^h, \tilde{y}^{f(h)}) \leq 0, \qquad (17)$$

$$p_s \sum_{h \in H} (x_s^h - y_s^{f(h)} - K_s(x_o^h - y_o^{f(h)}) - \omega_s^h) \leq \sum_{h \in H} \left(R_s \theta^h - D^h(\varphi^h, \tilde{y}^{f(h)}) \right). \qquad (18)$$

For n large enough, we must have $p_{ol} > 0, \forall l \in L$. Otherwise, every individual would choose $x_{ol}^h = n$ and we would have contradicted the first inequality of (14). But when $p_{ol} > 0$ we must have

$$\sum_H \left(x_{0l}^h - y_{0l}^{f(h)} - \omega_{0l}^h \right) = 0, \forall l \in L; \qquad (19)$$

since the aggregate budget constraint (17) of the first period is a null sum of nonpositive terms and therefore a sum of null terms.

Similarly, $\pi_j > 0$ for all $j \in J$. Otherwise, each individual would choose $\theta_j^h = n$, contradicting the second inequality of (14). But when $\pi_j > 0, \forall j \in J$, we must have

$$\sum_H \left(\theta^h - \varphi^h \right) = 0. \qquad (21)$$

This last contradiction is obtained because φ_j^h is bounded from above which is implied by the collateral constraint.

Combining (16) and the two last inequalities of (14) and using Lemma 2 and then Lemma 1, we have that the right side of (18) is zero. Hence,

$$p_s \sum_{h \in H} (x_s^h - y_s^{f(h)} - K_s(x_o^h - y_o^{f(h)}) - \omega_s^h) = 0$$

since the utility function is strictly increasing.

For n large enough, we must have $p_{sl} > 0, \forall l \in L$. Otherwise, every individual would choose $x_{sl}^h = n$ and we would have contradicted the inequality of (15). But when $p_{sl} > 0$ we must have

$$\sum_H \left(x_{sl}^h - \omega_{sl}^h - (K_s(x_o^h - y_o^{f(h)}))_l - y_{sl}^{f(h)} \right) = 0, \forall l \in L. \qquad (20)$$

It is useful to point out that the two contradictions obtained above are partly due to $H3$ which states that $y^{f(h)}$ is bounded from above.

- Step 4. Asymptotics of truncated equilibria. In this step, we will prove that the sequence of equilibria obtained in the previous step is a uniformly bounded sequence and therefore it has a convergent subsequence. Lastly, the limit of such subsequence will be the equilibrium for limit economy. In fact,
  let $\{(x_n^h, \theta_n^h, \varphi_n^h; y_n^{f(h)})_{h\in H}; (p_n, \pi_n); \overline{y}_n\}_{n\in Z_+}$ be the sequence of equilibria corresponding to $\mathcal{E}^n$. By definition, this sequence satisfies Items 1, 2, 3 and 4 of Definition 1

  1. From (14), it follows that

  $$\sum_{h\in H} x_{on}^h \leq \sum_{h\in H} (\omega_o^h + y_{on}^{f(h)})$$

     Non negativity of x_n^h and H3 with $v_o = \sum_{h\in H} \omega_o^h$ imply that the sequence $\{x_{on}^h\}_{n\in Z_+}$ is uniformly bounded.

     Since $-y_{on}^{f(h)}$ is non negative, it follows from (14) $-y_{on}^{f(h)} \leq \sum_{h\in H} \omega_o^h$ so that $\{y_{on}^{f(h)}\}_{n\in Z_+}$ is uniformly bounded. Using (15) and H3 with $v_s = \sum_{h\in H} \omega_s^h$ we have that the sequence $\{x_{ns}^h\}_{n\in Z_+}$ is uniformly bounded. To bind $y_{sn}^{f(h)}$, notice that the expression $\sum_{h\in H}(y_{sn}^{f(h)}+\omega_s^h) \geq 0$ and bounded from above from H3. Hence, non negativity of $y_{sn}^{f(h)}$ implies that the sequence $\{y_{sn}^{f(h)}\}_{n\in Z_+}$ is uniformly bounded.

     The collateral constraint (3), H4 and H3 imply that the sequence $\{\varphi_n^h\}_{n\in Z_+}$ of short sales is uniformly bounded. This fact and (21) imply that the sequence $\{\theta_n^h\}_{n\in Z_+}$ is also uniformly bounded. The price system (p_n, π_n, β_n) belongs to the simplex and therefore is uniformly bounded.

     It remains to prove that the sequence $\{\overline{y}_{sn}^j, j \in J\}_{n\in Z_+}$ is uniformly bounded. We state that if there is trading along the truncated economy, then the sequence $\{\overline{y}_{sn}^j, j \in J\}_{n\in Z_+}$ is bounded whenever $\sum_{h\in H} \varphi_{jn}^h \not\to 0$. Suppose that there is a subsequence $y_{sn_k} \to \infty$ as $k \to \infty$. Thus, the sequence $z_k := \overline{y}_{sn_k} \sum_{h\in H} \varphi_{jn_k}^h$ is unbounded. But from (16) it follows that the sequence z_k is bounded (because so is $\{\sum_{h\in H} p_{sn} y_{sn}^{f(h)}\}_{n\in Z_+}$) which is a contradiction. Hence, the earlier statement follows. Notice that this statement is independent of whether or not there is default. However, if all investors suffer default (i.e. $\mathcal{S}_s^j = H$ so that $\overline{y}_{sn}^j + p_{sn} K_S C_j < p_{sn} A_s^j$), the sequence $\{\overline{y}_{sn}^j\}_{n\in Z_+}$ is bounded regardless whether or not $\sum_{h\in H} \varphi_{jn}^h$ tends to zero.[8] Therefore the se-

[8] In fact, if $\overline{y}_{sn}^j + p_{sn} K_S C_j < p_{sn} A_s^j$, then $\overline{y}_{sn}^j < p_{sn} A_s^j$ since $p_{sn} K_s C_j \geq 0$. Now using the fact that p_{sn} belongs to the simplex, we have that $0 \leq \overline{y}_{sn}^j \leq ||A^j||$ where $||A^j|| = \max_{s\in S} |A_s^j|$.

quence $\{(x_n^h, \theta_n^h, \varphi_n^h; y_n^{f(h)})_{h\in H}; (p_n, \pi_n); \overline{y}_n\}_{n\in Z_+}$ has a subsequence converging to, say, $\big((x^h, \theta^h, \varphi^h; y^{f(h)})_{h\in H}; (p, \pi), \overline{y}\big)$

2. We state that the limit $\big((x^h, \theta^h, \varphi^h; y^{f(h)})_{h\in H}; (p, \pi), \overline{y}\big)$ is an equilibrium for the limit economy. Firstly, it satisfies Conditions 2 and 3 of Definition 1. This is obtained after taking the limit of the sub-sequence of truncated equilibria satisfying Conditions 2 and 3 of Definition 1. It remains to prove that for each h, the choice $(x^h, \theta^h, \varphi^h; y^{f(h)})$ is optimal given (p, π) and $\overline{y}$. Suppose that this is not the case. Then there exits $(x, \theta, \varphi; y) \in B^h(p, \pi, \overline{y})$ such that $U^h(x) > U^h(x^h)$.

   Lower hemicontinuity[9] of the budget correspondence $B^h(.)$ implies that there exists a subsequence $(x_n, \theta_n, \varphi_n; y_n)$ belonging to $B_n^h(p_n, \pi_n, \overline{y}_n)$ such that $(x_n, \theta_n, \varphi_n; y_n) \to (x, \theta, \varphi; y)$ when $n \to \infty$. Notice that the argument of $B^h(.)$ are the terms of the sequence that form part the sequence of equilibria of the truncated economy. That is, $(p_n, \pi_n, \overline{y}_n) \to (p, \pi, \overline{y})$. Since U^h is continuous, it follows that there exists $n_o \in Z_+$ such that

$$U^h(x_n) > U^h(x_n^h), \forall n \geq n_o$$

   contradicting the optimality of x_n^h in the truncated economy $\mathcal{E}^n$. Therefore, $(x^h, \theta^h, \varphi^h; y^{f(h)})$ is optimal given (p, π) and $\overline{y}$ in the limit economy.

3. Actually, the equilibrium whose existence was established is a non-trivial equilibrium. Following a similar argument to Araujo et al. (2005), suppose that assets are not traded, that is $\sum_{h\in H} \theta_j^h = 0$, then it is possible to set the endogenous average output level $\overline{y}_s^j$ such that it leads to a payment rate equal to 1. In fact, from (16), in the limit, it follows that the level $\overline{y}_s^j$ can be chosen large enough so that $\overline{y}_s^j + p_s K_s C_j > p_s A_s^j$ yielding $t_s^j = \min\{1, \frac{p_s K_s C_j + \overline{y}_s^j}{p_s A_s^j}\} = 1$.

[9]This property follows from Hildenbrand (1974, p. 26, fact 4). The proof is now standard in collateral models without production, see Araujo et al. (2000, 2005). For our case, the modifications are straightforward.

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