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Abstract

Short memory models contaminated by level shifts have similar long-memory features as fractionally integrated processes. This makes hard to verify whether the true data generating process is a pure fractionally integrated process when employing standard estimation methods based on the autocorrelation function or the periodogram. In this paper, we propose a robust testing procedure, based on an encompassing parametric specification that allows to disentangle the level shifts from the fractionally integrated component. The estimation is carried out on the basis of a state-space methodology and it leads to a robust estimate of the fractional integration parameter also in presence of level shifts. Once the memory parameter is correctly estimated, we use the KPSS test for presence of level shift. The Monte Carlo simulations show how this approach produces unbiased estimates of the memory parameter when shifts in the mean, or other slowly varying trends, are present in the data. Therefore, the subsequent robust version of the KPSS test for the presence of level shifts has proper size and by far the highest power compared to other existing tests. Finally, we illustrate the usefulness of the proposed approach on financial data, such as daily bipower variation and turnover.

Keywords: Long Memory, ARFIMA Processes, Level Shifts, State-Space methods, KPSS test.

JEL Classification: C10, C11, C22, C80.

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Non-technical Summary

The phenomenon of long memory has been known for years in fields like hydrology and physics. The hydrologist Hurst (1951) was the first to formally study that long periods of dryness of the Nile river were followed by long periods of floods. A formal theory on long memory processes was subsequently formulated by Mandelbrot (1975), who introduced the fractional Brownian motion and studied the so called *self-similarity* property. The introduction of fractional integration in economics and econometrics dates back to Granger (1980) and Granger and Joyeux (1980) who defined the autoregressive fractionally integrated moving average (ARFIMA) model. Similarly to hydrological and climatological time series, many economic and financial time series show evidence of being neither integrated of order zero (I(0)) nor integrated of order one (I(1) henceforth). In these circumstances the use of ARFIMA models becomes necessary. Nowadays, a broad range of applications in finance and macroeconomics shows that long-memory models are relevant - see among others Diebold et al. (1991) for exchange rate data, Andersen et al. (2001a) and Andersen et al. (2001b) for financial volatility series, and Baillie et al. (1996) for inflation data.

Short memory models contaminated by level shifts have similar long-memory features as fractionally integrated processes. This makes hard to verify whether the true data generating process is a pure fractionally integrated process when employing standard estimation methods based on the autocorrelation function or the periodogram. The literature on testing for fractional integration (*true* long-memory) versus *spurious* long-memory has grown in recent years. Mikosch and Starica (2004) test long-memory versus short-memory plus level shifts (or breaks) and propose a modified Dickey-Fuller test with shifts.

In this paper, we contribute to the literature of testing for fractional integration (*true* long-memory) versus *spurious* long-memory and we propose a robust testing procedure, based on an encompassing parametric specification that allows us to disentangle the level shifts from the fractionally integrated component. The estimation is carried out on the basis of a state-space methodology and it leads to a robust estimate of the fractional integration parameter also in presence of level shifts. Once the memory parameter is correctly estimated, we use the KPSS test for presence of level shift.

The Monte Carlo simulations show how this approach produces unbiased estimates of the memory parameter when shifts in the mean, or other slowly varying trends, are present in the data. Therefore, the subsequent robust version of the KPSS test for the presence of level shifts has proper size and by far the highest power compared to other existing tests. Finally, we illustrate the usefulness of the proposed approach on financial data, such as daily bipower variation and turnover. From the empirical analysis on a set of US stocks, it emerges that volatility and trading volume are likely to be characterized by the combined presence of both long-memory and level shifts.

1 Introduction

The phenomenon of long memory has been known for years in fields like hydrology and physics. The hydrologist Hurst (1951) was the first to formally study that long periods of dryness of the Nile river were followed by long periods of floods. A formal theory on long memory processes was subsequently formulated by Mandelbrot (1975), who introduced the fractional Brownian motion and studied the so called *self-similarity* property. The introduction of fractional integration in economics and econometrics dates back to Granger (1980) and Granger and Joyeux (1980) who defined the autoregressive fractionally integrated moving average (ARFIMA henceforth) model. Similarly to hydrological and climatological time series, many economic and financial time series show evidence of being neither integrated of order zero (I(0) henceforth) nor integrated of order one (I(1) henceforth). In these circumstances the use of ARFIMA models becomes necessary. Nowadays, a broad range of applications in finance and macroeconomics shows that long-memory models are relevant - see among others Diebold et al. (1991) for exchange rate data, Andersen et al. (2001a) and Andersen et al. (2001b) for financial volatility series, and Baillie et al. (1996) for inflation data. Early papers on the estimation of long-memory models are due to Fox and Taqqu (1986), Dahlhaus (1989), Sowell (1992) and Robinson (1995).

In order to carry out inference on the degree of long memory of a given time series, it is standard practice to look at the decay rate of its autocorrelation function or at its periodogram close to the origin. However, it is well known that the long memory feature can be generated by an I(0) process contaminated by levels shifts in the mean, see Diebold and Inoue (2001) and Granger and Hyung (2004). The so called *spurious* long-memory can arise from a short-memory process contaminated with level shifts, or by slowly varying trends. Mikosch and Starica (2004) stress that when a short memory process is contaminated by level shifts, its autocorrelation function mimics that of an ARFIMA process. Similarly, Dolado et al. (2008) show that the slow hyperbolic decay of the autocorrelation function, typical of the ARFIMA processes, could be confused with that generated by short memory processes whose mean is subject to breaks.

The literature of testing for fractional integration (*true* long-memory) versus *spurious* long-memory has grown in recent years. Mikosch and Starica (2004) test long-memory versus short-memory plus level shifts (or breaks) and propose a modified Dickey-Fuller test with shifts. Ohanissian et al. (2008) develop a test, based on temporal aggregation of the fractional process, that exploits the invariance of the fractional parameter to temporal aggregation. Shimotsu (2006) proposes two different strategies: the first one is based on sample splitting and subsequent comparison among different estimates of the memory parameter; the second one is based on a stationary test, such as KPSS and PP tests, performed on the d^{th}-differenced data. Perron and Qu (2010) propose a test based on log-periodogram regression with different bandwidths. Alternatively, Qu (2011) compares the spectral domain properties of long-memory and short-memory processes with level shifts at the critical frequency range. This score-type test is based on the derivatives of the profile local Whittle function and it does not require the specification of the shifting process. Xiaofeng and Xianyang (2010) propose a test to detect a mean shift with unknown dates in the time domain, which can be considered as a parametric version of Qu (2011). Recently, Leccadito et al. (2015) perform an extensive Monte Carlo exercise concluding that overall the test proposed by Qu (2011) has the best finite sample performance.

This paper proposes a new strategy to test whether an ARFIMA process is contaminated by random level shifts. With respect to the previous literature, our testing strategy is not based on the evaluation of the properties of a fractional process, but it is based on an encompassing specification in which both components, the ARFIMA and the random level shifts, are potentially present. In particular, we rely on a fully parametric approach, based on the state-space representation of the process, which allows obtaining robust estimates of the ARFIMA parameters as well as of the probability and the size of the random level shifts. Given the correct estimates of the model parameters, we can test for the absence of the shifting component. Thus, our approach can be considered as a robust version of the test proposed by Shimotsu (2006). Specifically, we use a two-step procedure: first, a modified Kalman filter routine, as proposed by Kim (1994), is adopted to estimate the model parameters by maximum likelihood; second, the null hypothesis of absence of the shift is tested by a KPSS statistic applied to the 'filtered' series (i.e. where the fractional component has been removed by d^{th}-difference).

The contributions of the paper can be summarized as follows: (i) we propose a state-space model to deal with the presence of both ARFIMA and random level shift processes; (ii) the corrected estimates of the ARFIMA parameters entail the possibility to compute a feasible and robust version of the KPSS test for the null hypothesis of absence of the level shifts; (iii) a Monte Carlo study shows that the test has the correct size and, in most cases, the highest power compared to the existing testing strategies; (iv) the testing procedure is robust to misspecifications of the shifting process and the empirical power is generally very high also when the ARFIMA is contaminated by a slowly moving trend; (v) the new testing method is carried out on a number of financial time series, such as daily bipower variation and turnover and the results are different from those obtained through other tests. In most cases, the results suggest that both ARFIMA and random changes in the mean are responsible for the long-run dependence in the data.

The paper is organized as follows. In Section 2, we specify the model as the sum of two unobserved components: an ARFIMA term and a level-shift process. Hence, we discuss the properties of the model, the corresponding state-space representation, the estimation methodology and the KPSS testing statistic. Section 3 reports the results of the Monte Carlo simulations, while Section 4 provides an empirical application and Section 5 concludes the paper.

2 ARFIMA model with level shifts

2.1 Model specification

Various semi-parametric approaches to test for *true* long memory, see among others Ohanissian et al. (2008), Perron and Qu (2010) and Qu (2011), focus on the properties that the series at hands must fulfill to be generated by a fractionally integrated process while the alternative hypothesis is not necessarily specified in a parametric form. In other words, a rejection of the null hypothesis of fractional integration is not informative on the properties of the data generating process (DGP henceforth) under the alternative.

Contrary to existing approaches, the methodology presented in this paper is based on the idea that the observed series may contain both the fractional and the level shift components. By doing so, we focus on whether the series is characterized by a level shift process - rather than looking at the properties that it must fulfill under the hypothesis of fractional integration. Using a fully parametric specification of the model, we obtain a robust estimate of the ARFIMA parameters both in presence and in absence of the contaminating term. Subsequently, we test for the presence of level shifts on the fractionally differenced series. This approach presents two advantages. First, the null and the alternative hypothesis are well defined in terms of the model parameters. Second, the presence of level shifts (or other slowly moving trends) does not automatically exclude the presence of a fractional component, but the two terms can co-exist.

We assume that the observed series is given by the sum of two unobserved components

$$y_t = \mu_t + x_t, \quad t = 1, \dots, T, \tag{1}$$

where x_t follows an ARFIMA(p, d, q) process and μ_t is the random shift component modeled as follows,

$$\mu_t = \mu_{t-1} + \delta_t, \quad \delta_t = \gamma_t \delta_{1t} + (1 - \gamma_t)\delta_{2t}, \tag{2}$$

where δ_t is a mixture of Gaussian random variables. In particular, $\delta_{jt} \sim \mathrm{N}(0, \sigma^2_{j\delta})$ for $j = 1, 2$, with $\sigma^2_{1\delta} = \sigma^2_\delta$ and $\sigma^2_{2\delta} = 0$, and the mixture is regulated by a Bernoulli random variable $\gamma_t \sim Bern(\pi)$. The ARFIMA(p, d, q) process x_t is defined as:

$$\Phi(L)(1 - L)^d x_t = \Theta(L)\xi_t, \quad t = 1, \dots, T, \tag{3}$$

where $\{\xi_t\}$ is a sequence of independent Gaussian random variables with zero mean and constant variance σ^2_ξ, the lag operator L is such that $Ly_t = y_{t-1}$; $\Phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$ is the autoregressive polynomial, while $\Theta(L) = 1 + \theta_1 L + \ldots + \theta_q L^q$, is the moving average operator, such that $\Phi(L)$ and $\Theta(L)$ have all their roots outside the unit circle, with no common factors. The long memory property is induced by the term $(1 - L)^d$, which is the fractional difference operator. The parameter d determines the fractional integration degree of the process, also known as memory parameter. If $d > -1/2$, x_t is invertible and has a linear representation. If $d < 1/2$, the process is covariance stationary. Furthermore, for $d > 0$ the process is said to have long memory and its autocorrelations die out at an hyperbolic rate (and indeed are no longer absolutely summable) in contrast to the (much faster) exponential rate for the weak dependence process. If $d = 0$ the process is an ARMA, also known as short memory process. The full vector of parameters of model (1) contains the ARFIMA parameters plus two extra parameters regulating the random level shift component, namely $\psi = (d, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \sigma_\xi^2, \sigma_\delta^2, \pi)$.

This formulation nests three different models: (i) when $\pi = 0$, y_t is a pure stationary ARFIMA process; (ii) for $\pi = 1$, μ_t is a random walk; (iii) if $\pi > 0$ then y_t is an ARFIMA process with random level shifts and the process y_t is non-stationary. Moreover, the non-stationarity degree of the process μ_t can be characterized in terms of summability order, see Berenguer-Rico and Gonzalo (2014).

Proposition 1. *Let the process μ_t be generated by model (2), with $\delta_{jt} \sim N(0, \sigma_{j\delta}^2)$ for $j = 1, 2$, with $\sigma_{1\delta}^2 = \sigma_\delta^2$ and $\sigma_{2\delta}^2 = 0$, and the mixture is regulated by a Bernoulli random variable $\gamma_t \sim Bern(\pi)$. It follows that*

$$\frac{1}{T^{3/2}} \frac{1}{\sigma_\delta \sqrt{\pi}} \sum_{t=1}^{[Tr]} \mu_t \xrightarrow{d} \int_0^r W(r) dr, \tag{4}$$

so that μ_t is summable of order 1, i.e. $\mu_t \sim I(1)$ process.

Proof in Appendix A.1.

The slowly varying function $L(T) = \frac{1}{\sigma_\delta \sqrt{\pi}}$ does not depend on time in this case and the convergence rate of μ_t is proportional to $\pi \sigma_\delta^2$. Hence, for a given σ_δ^2, the smaller π, the slower the convergence rate. When $\pi = 1$, we obtain the usual convergence rate of the random walk with i.i.d. innovations, i.e. $\frac{1}{T^{3/2}\sigma_\delta}$, with $L(T) = \frac{1}{\sigma_\delta}$. These results imply that y_t in (1) is I(1) when $\pi \cdot \sigma_\delta^2 > 0$, i.e. when the variance of the innovation of the shifting process μ_t and π are both non-zero. On the other hand, when $\pi \cdot \sigma_\delta^2 = 0$ then y_t is I(d). Indeed, the summability/integration order of x_t is given by $\frac{1}{T^{(d+0.5)}}$, so that x_t is summable of order d so that $x_t \sim$ I(d). Since the ARFIMA encompasses the class of ARMA processes, the specification in (1) nests also the case of a short memory ARMA process with $d = 0$ plus level shifts.

2.2 The testing procedure

We now outline a testing procedure for the absence of level shifts in model (1). Suppose that the true value of the fractional integration parameter, $0 \leq d_0 \leq 1/2$, is available, the observed series (1) can therefore be filtered as follows

$$\tilde{y}_t := (1 - L)^{d_0} y_t = \tilde{\mu}_t + \tilde{x}_t,$$

where

$$\Delta \tilde{\mu}_t = (1 - L)^{d_0} \delta_t, \quad \Phi(L) \tilde{x}_t = \Theta(L) \xi_t.$$

Under the null hypothesis $H_0 : \pi\sigma_\delta^2 = 0$, i.e. $\mu_t = 0 \ \forall t$, the filtered series is a stationary process, i.e. $\tilde{y}_t = \tilde{x}_t \sim I(0)$. Then we conclude that $y_t \sim I(d_0)$ process. Under the alternative hypothesis, $H_1 : \pi\sigma_\delta^2 > 0$, then y_t contains a level shift term, (i.e. $\mu_t \neq 0$), and the filtered series $\tilde{y}_t$ is non-stationary given that $\tilde{\mu}_t \sim I(1 - d_0)$.

A KPSS test statistic can be then computed as follows

$$\Psi = \frac{1}{T^2} \frac{\sum_{t=1}^{T} (\sum_{i=1}^{t} \tilde{y}_i)^2}{\tilde{\sigma}_y^2}, \tag{5}$$

where $\tilde{\sigma}_y^2$ is an estimate of the long-run variance of the filtered process $\tilde{y}_t$. Under the null hypothesis, the test has a Cramer-Von-Mises distribution, see Nyblom and Makelainen (1983), Leybourne and McCabe (1994) and Harvey (1997). Unfortunately, the test statistic in (5) is unfeasible since d_0 is unknown. In order to compute a feasible version of the KPSS test statistic, it is therefore necessary to obtain an unbiased estimate of the long memory parameter, $\hat{d}$, under both the null and the alternative hypothesis. This cannot be achieved with the traditional parametric and semiparametric estimators of the long memory parameter. Indeed, while those estimators are reliable and efficient under the null hypothesis, i.e. in absence of shifts, they are severely biased under the alternative. In the Section 2.3, we describe how to obtain reliable estimates of the long memory parameter that can be used to compute the feasible KPSS test statistic. Henceforth, the feasible test statistic, denoted by SSF_k, is

$$\Psi^* = \frac{1}{T^2} \frac{\sum_{t=1}^{T} (\sum_{i=1}^{t} \tilde{y}_i^*)^2}{\tilde{\sigma}_{y^*}^2}, \tag{6}$$

where $\tilde{y}_t^* = (1 - L)^{\hat{d}} y_t$ and $\tilde{\sigma}_{y^*}^2$ is its long run variance. The relevant quantiles of the asymptotic distribution of Ψ^* are tabulated in Shimotsu (2006). Interestingly, the quantiles of Ψ^* are very close to those of the standard Cramer-Von-Mises distribution when $0 < d_0 < 0.5$. The estimation method outlined below allows to correctly estimate $\hat{d}$ under the null and, more importantly, also under the alternative hypothesis. In other words, the proposed test methodology can be thought of as a robust version of the KPSS test of Shimotsu (2006), where the estimate of the memory parameter d is correct under both the null and the alternative hypothesis.

2.3 Estimation

We propose a robust estimation method of model (1) that relies on a modified Kalman filter routine as in Kim (1994). First, model (1) is cast in a state-space form

$$\begin{aligned} y_t &= Z\alpha_t, & & t = 1, \ldots, T, \\ \alpha_t &= F\alpha_{t-1} + R\eta_t, \qquad \eta_t \sim N(0, Q^{(j)}), & & j = 1, 2, \end{aligned} \tag{7}$$

where the state vector and the system matrices are defined as following

$$\alpha_t = (\mu_t, x_t, \ldots, x_{t-m+1})', \qquad \mathrm{Z} = (1, 1, 0, \ldots, 0), \qquad \eta_t = (\delta_t, \xi_t)',$$
$$\mathrm{F} = \begin{bmatrix} 1 & 0_{1\times m} \\ 0_{m\times 1} & \mathrm{F}_{22} \end{bmatrix}, \qquad \mathrm{R} = \begin{bmatrix} 1 & 0 \\ 0_{m\times 1} & \mathrm{R}_{22} \end{bmatrix}, \quad \mathrm{Q}^{(j)} = \begin{bmatrix} \sigma^2_{j\delta} & 0 \\ 0 & \sigma^2_{\xi} \end{bmatrix},$$
$$\mathrm{F}_{22} = \begin{bmatrix} \varphi_1 \cdots & \cdots \varphi_m \\ \mathrm{I}_{m-1} & 0_{(m-1)\times 1} \end{bmatrix}, \quad \mathrm{R}_{22} = (1, 0, \ldots, 0)'.$$

Note that the coefficients $\varphi_1, \ldots, \varphi_m$ are determined by the ARFIMA parameters, and $m > 0$ is the truncation order. Hosking (1981) shows that a stationary ARFIMA(p, d, q) admits infinite AR (and MA) expansions and provides a formula to compute the coefficients of such representations as an infinite convolution of the AR (and MA) filter with the fractional difference operator. Although a long memory process has infinite AR and MA representations, Chan and Palma (1998) propose an approximation based on the truncation up to the m-th lag and provide the asymptotic properties of the truncated ML estimator. The small sample properties of state space approach has been recently investigated by Grassi and Santucci de Magistris (2014), who have shown the reliability of the methodology. In particular, they show that the ML estimator based on the truncated model is consistent, asymptotically Gaussian and efficient. For details on the state-space method for long-memory process see Appendix A.2.

Define $\mathrm{S}_{t-1} = i$ as the state at time $t-1$ and $\mathrm{S}_t = j$ as the state at time t, with $i, j = 1, 2$. Using the pair $\{i, j\}$ the first element denotes the *past regime* and the second one refers to the *present regime*. We define with $\mathrm{Y}_t = \{y_t, \ldots, y_1\}$, the information set up to time t, with $\pi^{(i,j)}_{t|t} := \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j|\mathrm{Y}_t)$ the *real-time* filter probability to switch from i to j, and the *real-time* filter probability to be in j is then equal to $\pi^{(j)}_{t|t} := \Pr(\mathrm{S}_t = j|\mathrm{Y}_t) = \sum_{i=1}^2 \pi^{(i,j)}_{t|t}$. Similarly, the *predictive* filter probability to switch from i to j, is $\pi^{(i,j)}_{t|t-1} := \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j|\mathrm{Y}_{t-1})$, and the *predictive* filter probability to be in state j' is $\pi^{(j)}_{t|t-1} := \Pr(\mathrm{S}_t = j|\mathrm{Y}_{t-1}) = \sum_{i=1}^2 \pi^{(i,j)}_{t|t-1}$. While the constant *transition probability* is $\lambda_{ij} := \Pr(\mathrm{S}_t = j|\mathrm{S}_{t-1} = i) = \Pr(\mathrm{S}_t = j) = \lambda_j$, with $\lambda_1 = \pi$ and $\lambda_2 = (1 - \pi)$, and it will one of the static parameter to be estimated.

The representation (7) differs from the standard state-space representation of an ARFIMA model because the matrix $\mathrm{Q}^{(j)}$ is subject to stochastic changes driven by the shift parameters π and σ^2_δ. Therefore, the Kalman filter routine required to compute the log-likelihood function of the model needs to be modified according to Kim (1994) and Kim and Nelson (1999). Differently from the specification adopted in Grassi and Santucci de Magistris (2014), the measurement equation links the levels of the observed variable y_t to the unobserved states. When working with the first difference of y_t as in Grassi and Santucci de Magistris (2014), the innovation to the shift term, δ_t, enters directly in the measurement equation and it is treated as a measurement error. Instead, in the above representation both the ARFIMA term and the level shift are modeled as unobserved state variables and the variances of their innovations enter in the matrix $\mathrm{Q}^{(j)}$. In this way, the measurement equation can be extended with the inclusion of other unobserved components, if necessary.

The predictive filter for the state vector and its mean squared error (MSE) are

$$\tilde{\alpha}_{t|t-1}^{(i,j)} = \mathrm{F}\tilde{\alpha}_{t-1|t-1}^{(i)}, \qquad \mathrm{P}_{t|t-1}^{(i,j)} = \mathrm{F}\mathrm{P}_{t-1|t-1}^{(i)}\mathrm{F}' + \mathrm{R}\mathrm{Q}^{(j)}\mathrm{R}', \tag{8}$$

where $\tilde{\alpha}_{t|t-1}^{(i,j)} = \mathrm{E}(\alpha_t|\mathrm{Y}_{t-1}, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$ and $\mathrm{P}_{t|t-1}^{(i,j)} = \mathrm{Var}(\alpha_t|\mathrm{Y}_{t-1}, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$. The corresponding prediction error and its MSE are

$$v_t^{(i,j)} = y_t - \mathrm{Z}\tilde{\alpha}_{t|t-1}^{(i,j)}, \qquad \mathrm{G}_t^{(i,j)} = \mathrm{Z}\mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'. \tag{9}$$

The real-time filter and its MSE for the transition state are

$$\tilde{\alpha}_{t|t}^{(i,j)} = \tilde{\alpha}_{t|t-1}^{(i,j)} + [\mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'/\mathrm{G}_t^{(i,j)}]v_t^{(i,j)}, \qquad \mathrm{P}_{t|t}^{(i,j)} = \mathrm{P}_{t|t-1}^{(i,j)} - \mathrm{P}_{t|t-1}^{(i,j)}\mathrm{Z}'\mathrm{Z}\mathrm{P}_{t|t-1}^{(i,j)}/\mathrm{G}_t^{(i,j)}. \tag{10}$$

where $\tilde{\alpha}_{t|t}^{(i,j)} = \mathrm{E}(\alpha_t|\mathrm{Y}_t, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$ and $\mathrm{P}_{t|t}^{(i,j)} = \mathrm{Var}(\alpha_t|\mathrm{Y}_t, \mathrm{S}_{t-1} = i, \mathrm{S}_t = j)$. Given real-time filter probability to be in state "i" at time $t-1$, that is $\pi_{t-1|t-1}^{(i)}$, we can compute the predictive filter transition probability

$$\pi_{t|t-1}^{(i,j)} = \lambda_j \pi_{t-1|t-1}^{(i)}, \tag{11}$$

and the predictive filter probability to be in state "j", is obtained as follows $\pi_{t|t-1}^{(j)} = \sum_{i=1}^{2} \pi_{t|t-1}^{(i,j)}$. The conditional probability of the observation is obtained as a weighted average of the single conditional Gaussian probabilities

$$\Pr(y_t|\mathrm{Y}_{t-1}) = \sum_{i=1}^{2}\sum_{j=1}^{2} \Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}, \tag{12}$$

where the observations are conditionally Gaussian

$$\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1}) = \left[2\pi\mathrm{G}_t^{(i,j)}\right]^{-1/2} \exp\left[-\frac{v_t^{(i,j)2}}{2\mathrm{G}_t^{(i,j)}}\right]. \tag{13}$$

Given the predictive filter transition probabilities, we can now update the real-time filter transition probabilities

$$\pi_{t|t}^{(i,j)} = \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}}{\Pr(y_t|\mathrm{Y}_{t-1})}, \tag{14}$$

and we can aggregate them to obtain the real-time filter probability to be in state "j" which is $\pi_{t|t}^{(j)} = \sum_{i=1}^{2} \pi_{t|t}^{(i,j)}$. Using (11)-(14) we can then construct the full log-likelihood function

$$\ell(\mathrm{Y}_T|\psi) = \sum_{t=1}^{T} \log\left[\Pr(y_t|\mathrm{Y}_{t-1})\right], \tag{15}$$

where ψ is the set of unknown parameters. Finally, the real-time filter for the state vector to be in state "j" and its MSE are

$$\begin{aligned}\tilde{\alpha}_{t|t}^{(j)} &= \left[\sum_{i=1}^{2} \pi_{t|t}^{(i,j)} \tilde{\alpha}_{t|t}^{(i,j)}\right] / \pi_{t|t}^{(j)}, \\ \mathrm{P}_{t|t}^{(j)} &= \left[\sum_{i=1}^{2} \pi_{t|t}^{(i,j)} \{\mathrm{P}_{t|t}^{(i,j)} + [\tilde{\alpha}_{t|t}^{(j)} - \tilde{\alpha}_{t|t}^{(i,j)}][\tilde{\alpha}_{t|t}^{(j)} - \tilde{\alpha}_{t|t}^{(i,j)}]'\}\right] / \pi_{t|t}^{(j)},\end{aligned} \tag{16}$$

and the final filter estimate is

$$\tilde{\alpha}_{t|t} = \sum_{j=1}^{2} \pi_{t|t}^{(j)} \tilde{\alpha}_{t|t}^{(j)}.$$

To summarize, for $t = 1, \dots, T$, we compute the set of Kalman filter recursions (8)-(10) and (16). In parallel, using (11)-(15) we compute the probabilities and the log-likelihood function which is then maximize with respect to the vector of parameters ψ. For more details and derivations see Appendix A.3.

The recursions in (8) are initialized as follows: the first element of the state vector with "diffuse prior" for $i = 1$ (see Harvey (1989), sec. 3.3.4), and with a constant for $i = 2$. The remaining m elements are initialized with the unconditional mean and variance of the stationary long-memory process. Namely,

$$\tilde{\alpha}_{0|0}^{(i)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathrm{P}_{0|0}^{(i)} = \begin{bmatrix} \mathrm{P}_{\mu}^{(i)} & 0 \\ 0 & \mathrm{P}_{x}^{(i)} \end{bmatrix}, \quad i = 1, 2,$$

with $\mathrm{P}_{\mu}^{(1)} = \kappa$, with $\kappa \to \infty$, and $\mathrm{P}_{\mu}^{(2)} = 0$, while $\mathrm{P}_{x}^{(1)} = \mathrm{P}_{x}^{(2)}$ is initialized as described in Appendix A.2. Similarly, the recursion in (11) is initialized as follows: $\pi_{0|0}^{(1)} = \pi$ and $\pi_{0|0}^{(2)} = 1 - \pi$.

3 Monte Carlo analysis

We perform a set of Monte Carlo simulations to evaluate the finite-samples ability of the proposed testing methodology. We study the empirical size and power of the test based on alternative choices of the parameters governing the ARFIMA process x_t and the shifting process μ_t. The proposed method presents the potential pitfall of being fully parametric, meaning that model misspecification may induce severe size distortions and power losses. Hence, the Monte Carlo simulations are carried out to evaluate the robustness of the estimation method to misspecification in μ_t, for which we follow the DGPs used in Qu (2011, p.430).

The performance of the proposed procedure is assessed relatively to several existing tests. In particular, we consider Ohanissian et al. (2008) (ORT, henceforth), Perron and Qu (2010) (PQ, henceforth), Qu (2011) (QU, henceforth), and the three tests of Shimotsu (2006): SH_k and SH_p based on KPSS and PP test statistic respectively, and the SH_s based on the sample splitting.[1] Since the ARMA dynamics worsen the finite sample properties of the Qu (2011) test, this is performed using the *pre-whitening* procedure outlined in Qu (2011) to remove

[1]Due to space constraints, we don't provide a detailed discussion of these tests. A formal presentation of all these tests can be found in Leccadito et al. (2015).

the ARMA terms and keep the empirical size of the test under control. As described in Qu (2011, p.429), a low-order ARFIMA(p,d,q) with $p, q \leq 1$ is fitted on the observed series, y_t, and the optimal ARMA lag-order is determined with the Akaike information criterion (AIC). Subsequently, the series y_t^*, i.e. y_t filtered from the ARMA terms, is used to construct the test statistics. Notice that our approach does not need this *prewhitening* procedure, since all the parameters are jointly estimated as outlined above. In the next section, we present the results for the empirical size, the empirical power and the case in which both ARFIMA and shifts components are present in the data.

3.1 Empirical size

The empirical size of the test is assessed when y_t does not contain any level shifts, i.e. $\pi \cdot \sigma_\delta^2 = 0$, which in turns implies that $\mu_t = 0\ \forall t \in [0, T]$. In this case, y_t is a stationary ARFIMA process, $(1 - \phi L)^d y_t = (1 - \theta L)\xi_t$, $\xi_t \sim \text{iiN}(0, \sigma_\xi^2)$, with $d = 0.4$ and different combinations of the ARMA terms, as in Qu (2011). The sample sizes are $T = 500, 1000, 2000$. Table 1 reports the empirical rejection frequencies at 5% nominal level of the various test statistics considered. The last column reports the Monte Carlo average, based on $M = 1000$ replicates, of the estimates of the fractional parameter, d, based on the state-space representation. It emerges that the estimates of the long memory parameter are very close to the true value in all cases.[2] In other words, the modified Kalman filter routine is able to provide the proper estimates of the ARFIMA parameters also when the level shift process is absent. Indeed, the estimates of π and σ_δ^2 are such that their product is very small, meaning that the estimated shift component is negligible if compared to the ARFIMA component. This makes the empirical size of the SSF_k test very close to the nominal value, with a slight over-rejection rate for small sample. The other reported tests have also good size properties. In general, the Qu (2011) test is slightly conservative in small samples as well in large samples. Overall, the empirical size of the proposed testing strategy is in line with that of the other tests and in line with the findings in Qu (2011) and Leccadito et al. (2015).

3.2 Empirical power

The major advantage of the proposed procedure arises when looking at the empirical power of the test, i.e. $\mu_t \neq 0$. In line with Qu (2011, p.430), we consider a signal, x_t, contaminated by a trend process, μ_t, i.e. $y_t = x_t + \mu_t$. In order to assess the robustness of the SSF_k test to different trend processes, the estimation/testing procedure is carried out under the following DGPs for μ_t:

1. **Non-stationary random level shifts**: $\mu_t = \mu_{t-1} + \gamma_t \delta_t$, with $\delta_t \sim \text{N}(0, 1)$, $\gamma_t \sim Bern(6/T)$;

2. **Stationary random level shifts**: $\mu_t = (1 - \gamma_t)\mu_{t-1} + \gamma_t \delta_t$, with $\delta_t \sim \text{N}(0, 1)$, $\gamma_t \sim Bern(0.003)$;

[2] We do not report the Monte Carlo average of the estimates of the other model parameters but they are available upon request to the authors.

3. **Monotonic trend**: $\mu_t = 3t^{-0.1}$;

4. **Non-monotonic trend**: $\mu_t = \sin(4\pi t/T)$;

5. **Markov-switching**: $\mu_t \sim \mathrm{N}(1,1)$ if $s_t = 0$ and $\mu_t \sim \mathrm{N}(-1,1)$ if $s_t = 1$, with state transition probabilities $\pi_{10} = \pi_{01} = 0.001$;

6. **Markow-switching with GARCH regimes**: $\mu_t = \log(r_t^2)$ with $r_t = \sqrt{h_t} z_t$ with $h_t = 1 + 2s_t + 0.4r_{t-1}^2 + 0.3h_{t-1}$, and $z_t \sim \mathrm{N}(0,1)$ with $\pi_{10} = \pi_{01} = 0.001$.

Note that the first specification for μ_t matches the DGP in equation (2). However, the finite sample properties of the SSF_k test are also investigated for other types of trends that can characterize the observed series. In addition to the setup in Qu (2011), where an i.i.d. Gaussian random variable, x_t, is added to the shifting process, μ_t, we also consider the possibility that x_t follows an ARFIMA process. In our setup, the two sources of persistence can co-exist and the state space approach is designed to disentangle them, thus providing reliable parameter estimates in both cases.

3.2.1 Non-stationary random level shifts

The main result that emerges from Table 2 is that the proposed testing procedure has very high power for all the cases considered and for all sample sizes, as opposed to most existing tests. This result is mainly due to the ability of our method to provide good estimates of the model parameters when the observed data, y_t, is given by the sum of a non-stationary random level shift process, μ_t, and an ARFIMA process, x_t, with different parameter combinations. The highest empirical power is obtained when a white noise process is added to μ_t, which is the case considered in Qu (2011). In particular, when $T = 2000$, the empirical rejection rate of the null hypothesis obtained by the SSF_k is almost 100% and the estimates of d are centered around 0, as expected. The highest power obtained by the other tests is that of the KPSS statistic of Shimotsu (2006) which in any case falls well below 100%. The empirical power of the SSF_k remains high also when ARFIMA processes with $d = 0.2$ are considered for x_t. In all cases, the estimates of d are centered around the true value, confirming the validity of the state space approach when both the long memory component and the shifts are present. The power is drastically reduced when we consider a highly persistent ARFIMA process with $\phi = 0.8$. Indeed, as indicated by the variance ratio (VR, henceforth) reported in the last column of the table, the variability of the shift process relative to the total variability of y_t is only one third of that of the white-noise case. It follows that it is relatively more difficult to conduct precise inference on the shift process when the ARFIMA series is more persistent, and this impacts on the empirical power of the SSF_k test. Moreover, as noted by Grassi and Santucci de Magistris (2014), the estimates of the parameter d become more imprecise as the AR parameter gets closer to 1. It should be noted that this parameter configuration is rather extreme and not often found in the real data. Interestingly, the power of the SSF_k remains the highest also in this case, while the power of the other semi-parametric tests is almost null for all sample sizes.

3.2.2 Stationary random level shifts

The good performance of the SSF_k test is confirmed also when an ARFIMA process is contaminated by a stationary random level shift process, see Table 3. The power of the SSF_k test in detecting the presence of the shift process is the highest in almost all cases considered. This is again due to our approachs ability to provide accurate parameter estimates in all cases. Indeed, the estimates of the long memory parameter d, reported in the last column of the table, are always centred around the true value. Also in this case, we note a relatively low empirical power for the ARFIMA with $\phi = 0.8$ as a consequence of the low VR and the biased estimates of the long memory parameter. However, the empirical power of the SSF_k remains the highest compared to the semi-parametric alternatives.

3.2.3 Monotonic and non-monotonic trends

The SSF_k test performs surprisingly well also when non-stochastic trends are present in the data, see Tables 4 and 5. Although the model specification is not designed to account for those DGPs, our method provides a good tracking of the deterministic trends when they are present in the data. Thus, the empirical power of the SSF_k test is almost 1 in many cases and it drops only when a highly persistent ARFIMA process is present in the data. Relatively to the other semi-parametric tests, the power of our test is extremely high for the monotonic trend. For the non-monotonic trend, we observe a good performance of the Qu (2011) test, with the exclusion of the ARFIMA(1,0.2,0) with $\phi = 0.8$. Interestingly, the power of the SSF_k is high even though the VR is relatively low compared to the random level-shift processes in the previous tables.

3.2.4 Markov-switching processes

Tables 6 and 7 report the results for a trend μ_t generated by a Markow-switching process. Also in this case, the observed series is obtained by the sum of the Markov-switching process and an ARFIMA term. The power of the SSF_k test is generally the highest compared to the methods and the estimates of d are well centred on the true values with the exception of the case with $\phi = 0.8$. In this case, the best performance is achieved by the KPSS test of Shimotsu (2006), but the power is rather low for all T. On the contrary, the power of the SSF_k relative to the other tests is very high when the ARFIMA process contains MA terms or when x_t is an i.i.d. noise for moderate values of T.

When the Markov switching term is characterized by GARCH effects, the overall power is rather low for all tests. Still, the SSF_k has the overall best performance, but the power is above 50% only in one case, when x_t follows an ARFIMA with $\theta = 0.8$ and $T = 2000$. In fact, the estimates of the fractional parameter d are generally far from the true value and this leads to lower power than that achieved under different DGPs for μ_t. This result is mainly due to the fact that, although the VR is rather high in this case, the autocorrelation function of μ_t is not characterized by very high persistence, so that it is hard for the modified Kalman filter of Kim (1994) to disentangle μ_t and x_t, with consequent power losses.

4 Empirical application

Let us now apply the SSF_k test to a number of financial time series for which evidence of fractional integration has been documented. In particular, we choose daily bipower-variation and turnover, which is the trading volume divided by the number of outstanding shares. The sample covers 15 stocks traded on NYSE for the period between January 2, 2003 and June 28, 2013, for a total of 2640 observations. As it has been widely shown in the past, the series of realized volatility and bipower-variation are characterized by long-range dependence, or long memory, see Andersen et al. (2001a), Andersen et al. (2001b), Andersen et al. (2003), Andersen et al. (2009) and Martens et al. (2009). Analogously, it has been documented that trading volume also displays long-range dependence features. For instance, Bollerslev and Jubinski (1999) and Lobato and Velasco (2000) both report strong evidence that volume exhibits long memory. More recently, Rossi and Santucci de Magistris (2013) study the common dynamic dependence between volatility and volume and find evidence of fractional cointegration only for the series belonging to the bank/financial sector, i.e. those that during the financial crises have experienced a large upward level shift. It is therefore of interest to be able to formally test, although in an univariate setup, if volatility and volume are subject to level shifts or if their long-run dependence is more likely generated by a pure fractional process.

The bipower-variation is constructed using log-returns at 1-minute frequencies as

$$BPV_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{j=2}^{M}|r_{t,j-1}|\cdot|r_{t,j}|, \tag{17}$$

where $r_{t,j}$ is the j-th log-return on day t and $M = 390$ is the number of intra-daily observations associated to 1-minute frequencies. The BPV_t estimator converges to the daily integrated variance, i.e. the instantaneous variance cumulated over daily horizons, and it is robust to price jumps. For what concerns the daily turnover, they are defined as

$$TRV_t = \frac{V_t}{S_t}, \tag{18}$$

where V_t is the trading volume, i.e. number of shares that have been bought and sold within day t and S_t is the number of shares available for sale by the general trading public at time t. The turnover is by construction more robust than trading volume to effects like stock splits and it does not display large upward trends as V_t. The empirical analysis is carried out on the log-transformed series, $\log(BPV_t)$ and $\log(TRV_t)$ as the model (1) involves unobserved components that are defined on the entire set of real numbers. Moreover, although the distributions of bipower-variation and turnover are clearly right-skewed, the distributions of their logarithms are approximately Gaussian.

Tables 8 and 9 report the values of the tests for the presence of fractional integration in $\log(BPV_t)$ and $\log(TRV_t)$. For what concerns $\log(BPV_t)$, the SSF_k rejects the null hypothesis of absence of shift for 8 out of 15 stocks at 5% significance level. Interestingly, the highest values of the tests are associated with the companies operating in the financial sector, like

Bank of America (BAC), Citygroup (C), JP-Morgan (JPM) and Wells Fargo (WFC). These companies have been subject to a major financial distress during the 2008-2009 financial crisis, and the values of BPV_t have been extremely high for many months in this period. The test of Perron and Qu (2010) also seems to find significant evidence of shifts for three out of four volatility series of the stocks in the bank sector. However, the tests based on semi-parametric specifications are unable to reject the null hypothesis of fractional integration in most cases. This may be a consequence of their rather low power, as it emerged in the Monte Carlo study. Indeed, looking at the semi-parametric estimates of d, obtained with the Whittle estimator, they lie generally above the stationary threshold, i.e. $\hat{d} > 0.5$. On the other hand, the estimates obtained with the state-space methodology are still positive but significantly smaller than 0.5 (with the exception of PG), meaning that a large portion of the observed long-run dependence is attributed to the random level shifts (or possibly other trend components).

For what concerns $\log(TRV_t)$, the SSF_k rejects the null hypothesis of absence of shifts for 13 out of 15 stocks at 5% significance level, and the estimated fractional parameter is significantly larger than 0 in all cases. Interestingly, there is an almost unanimous agreement across all tests that the turnover series of BA, HPQ, JPM and PEP present spurious long-memory features, while the assumption of truly long-memory for the $\log(TRV_t)$ of PG is only rejected by the SH_s test. Again, the highest values of the SSF_k test are associated with BAC, C, JPM and WFC. This seems to provide some evidence in favour of the long run relationship between volatility and volume as possibly driven by the joint presence of shifts and not only by a common fractional trend. Indeed, both $\log(BPV_t)$ and $\log(TRV_t)$ may be generated by the combination of a fractional process and a shift (or a potentially non-linear and smooth trend). Thus, the current univariate setup could be extended to the multivariate case and, in particular, to the possibility of jointly model common fractional trends and common shifts. This extension, coupled with the definition of an efficient method to track the shifting process given the parameter estimates, is left to future research.

5 Conclusion

In this paper, we have proposed a robust testing strategy for a fractional process potentially subject to structural breaks. Contrary to the other tests for true fractional integration presented so far in the literature, the focus of our approach is on the level shift process. We propose a flexible state-space parametrization that is able to account for the presence of both an ARFIMA and a level-shift process. In particular, our parametric approach provides robust estimates of the memory parameter for a given time series possibly subject to level shifts or other smoothly varying trends. The testing procedure can be seen as a robust version of the KPSS test for the presence of level shifts. A Monte Carlo study shows that the proposed method performs much better than the other existing tests, especially under the alternative. Interestingly, the modified Kalman filter routine adopted to estimate the model parameters is robust to a variety of different contamination processes and it is reliable also when slowly varying trends characterize the data. From the empirical analysis on a set of US stocks, it emerges that volatility and trading volume are likely to be characterized by the combined presence of both long-memory and level shifts. This result differs from that of other existing tests, which usually over-estimate the long memory parameter and are characterized by low power. The theoretical and empirical results outlined in this paper call for extensions in several directions. For example, the tracking of the shift process, possibly using a smoothing algorithm, would be very informative on the type of trend that characterizes the data and could be exploited for forecasting purposes. Alternatively, a multivariate extension of model (1) would allow to distinguish and test the hypothesis of fractional cointegration in a context characterized by common and idiosyncratic level shifts.

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A Appendix

A.1 Summability order of the level shift component μ_t

The summability condition of the process μ_t can be derived by looking at μ_t as a random walk, where the innovation is $z_t = \gamma_t \delta_t$, where $\gamma_t \sim Bern(\pi)$ and $\delta_t \sim \mathrm{N}(0, \sigma_\delta^2)$, so that

$$\mathrm{E}(z_t) = E(\gamma_t) \cdot E(\delta_t) = \pi \cdot 0 = 0, \tag{19}$$

$$\begin{aligned} \mathrm{Var}(z_t) &= \mathrm{Var}(\gamma_t) \cdot \mathrm{Var}(\delta_t) + \mathrm{Var}(\delta_t) \cdot \mathrm{E}(\gamma_t)^2 + \mathrm{Var}(\gamma_t) \cdot E(\delta_t)^2 \\ &= p(1-\pi)\sigma_\delta^2 + \pi^2 \sigma_\delta^2 + 0 = \pi \cdot \sigma_\delta^2. \end{aligned} \tag{20}$$

Therefore the partial sum

$$\frac{1}{T^{\frac{1}{2}}} \frac{1}{\sigma_\delta \sqrt{\pi}} \sum_{t=1}^{[Tr]} z_t \xrightarrow{d} W(r), \tag{21}$$

so that $z_t \sim S(0)$ and $z_t \sim \mathrm{I}(0)$. Provided that $\mu_t = \sum_{t=1}^{T} z_t$, then

$$\frac{1}{T^{\frac{3}{2}}} \frac{1}{\sigma_\delta \sqrt{\pi}} \sum_{t=1}^{[Tr]} \mu_t \xrightarrow{d} \int_0^r W(r)dr, \tag{22}$$

so that $\mu_t \sim S(1)$ and $\mu_t \sim \mathrm{I}(1)$, but with a slowly varying function equal to $L(T) = \frac{1}{\sigma_\delta \sqrt{\pi}}$, see Berenguer-Rico and Gonzalo (2014). When $\pi = 1$ we obtain the usual convergence rate of the random walk, with i.i.d. innovations. It is clear that the convergence depends on π, so that to smaller π corresponds a slower convergence rate.

A.2 Estimation of ARFIMA models by state-space methods

Here we recall how to estimate the ARFIMA model introduced in Section 1 using state-space methods. Following Harvey (1989) and Harvey and Proietti (2005), the *time invariant* state space representation consists of two equations. The first is the *measurement equation*, which relates the univariate time series, y_t, to the state vector:

$$y_t = \mathrm{Z}\alpha_t + \varepsilon_t, \qquad t = 1, ..., T, \tag{23}$$

where Z is $1 \times m$ selection vector, α_t is $m \times 1$ state vector with initial values $\alpha_1 \sim \mathrm{N}(\tilde{\alpha}_{1|0}, \mathrm{P}_{1|0})$ and $\varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon^2)$ is the measurement error. The second is the *transition equation*, that defines the evolution of the state vector α_t as a first order vector autoregression:

$$\alpha_t = \mathrm{F}\alpha_{t-1} + \mathrm{R}\eta_t, \qquad \eta_t \sim \mathrm{N}(0, \mathrm{Q}), \tag{24}$$

where F is $m \times m$ matrix, R is $m \times g$ selection matrix, and η_t is a $g \times 1$ disturbance vector and Q is a $g \times g$ variance-covariance matrix. The two disturbances are assumed to be uncorrelated

$E(\varepsilon_t \eta'_{t-j}) = 0$ for $j = 0, 1, ..., T$.

Let define $Y_t = \{y_1, \dots, y_t\}$ as the information set up to time t, the state vector α_t and the observations y_t, are conditional Gaussian, i.e. $\alpha_t | Y_{t-1} \sim N(\tilde{\alpha}_{t|t-1}, P_{t|t-1})$ and $y_t | Y_{t-1} \sim N(Z\tilde{\alpha}_{t|t-1}, F_t)$, with mean and variance computed by the Kalman filter (KF) recursions

$$\begin{aligned} v_t &= y_t - Z\tilde{\alpha}_{t|t-1}, \quad t = 1, \dots, T, \\ G_t &= ZP_{t|t-1}Z' + \sigma^2_\varepsilon, \\ K_t &= FP_{t|t-1}Z'G_t^{-1}, \\ \tilde{\alpha}_{t+1|t} &= F\tilde{\alpha}_{t|t-1} + K_t v_t, \\ P_{t+1|t} &= FP_{t|t-1}F' - K_t G_t K'_t + RQR'. \end{aligned} \tag{25}$$

The algorithm is initialized with the unconditional mean $\tilde{\alpha}_{1|0} = 0$ and the unconditional variance $\text{vec}(P_{1|0}) = (I - F \otimes F)^{-1}\text{vec}(RQR')$. The system matrices are deterministically related with the vector of parameter ψ, thus we can construct the log-likelihood function:

$$\ell(Y_T; \psi) = \sum_{t=1}^{T} \log \Pr(y_t | Y_{t-1}; \psi) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} \log G_t - \frac{1}{2}\sum_{t=1}^{T} \frac{v_t^2}{G_t}. \tag{26}$$

In case we are interested in the "contemporaneous filter" or "real-time estimate" of the state vector, we have that $\alpha_t | Y_t \sim N(\tilde{\alpha}_{t|t}, P_{t|t})$, where

$$\begin{aligned} \tilde{\alpha}_{t|t} &= \tilde{\alpha}_{t|t-1} + P_{t|t-1}Z'G_t^{-1}v_t, \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}Z'G_t^{-1}ZP_{t|t-1}. \end{aligned} \tag{27}$$

Looking at equations (25) and (27), we can notice that the filtering (25) can be obtained from (27) as follows

$$\begin{aligned} \tilde{\alpha}_{t+1|t} &= F\tilde{\alpha}_{t|t}, \\ P_{t+1|t} &= FP_{t|t}F' + RQR'. \end{aligned} \tag{28}$$

Equations (27) together with the prediction error v_t and its variance G_t are known as the "updating step", while the equations (28) are known as the "prediction step". To derive the set of recursions for the model with switching parameters we break down the filtering in those two steps.

The ARFIMA model (3) has the following autoregressive (AR) representation $\varphi(L)x_t = \xi_t$, where

$$\varphi(L) = 1 - \sum_{j=1}^{\infty} \varphi_j L^j = (1-L)^d \frac{\Phi(L)}{\Theta(L)}, \quad (1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} L^j,$$

Its moving average (MA) representation is $x_t = \Psi(L)\xi_t$, where

$$\zeta(L) = 1 + \sum_{j=1}^{\infty} \psi_j L^j = (1-L)^{-d} \frac{\Theta(L)}{\Phi(L)}, \quad (1-L)^{-d} = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} L^j.$$

Chan and Palma (1998) show that the exact likelihood function is obtained using the AR (or MA) representation of order T. In order to make the state-space methods feasible, they propose to a truncation up to lag m. In particular, the truncated AR(m) representation is

$$\begin{aligned}
&\alpha_t = (x_t, \ldots, x_{t-m+1})', \quad \mathrm{Z} = (1,0,...,0), \quad \sigma^2_\varepsilon = 0,\\
&\mathrm{F} = \begin{bmatrix} \varphi_1 \cdots & \varphi_m \\ \mathrm{I}_{m-1} & 0 \end{bmatrix}, \quad \mathrm{R} = (1,0,...,0)', \quad \mathrm{Q} = \sigma^2_\xi.
\end{aligned} \tag{29}$$

Similarly, the truncated MA(m) representation is

$$\begin{aligned}
&\alpha_t = (x_t, \tilde{x}_{t|t-1}, \ldots, \tilde{x}_{t+m-1|t-1})', \qquad \mathrm{Z} = (1,0,...,0), \qquad \sigma^2_\varepsilon = 0,\\
&\mathrm{F} = \begin{bmatrix} 0 & \mathrm{I}_m \\ 0 & 0' \end{bmatrix}, \qquad \mathrm{R} = (1, \zeta_1, \ldots, \zeta_m)', \quad \mathrm{Q} = \sigma^2_\xi.
\end{aligned} \tag{30}$$

The ML estimator, $\hat{\psi} = \arg\max \ell(\mathrm{Y}_T; \psi)$, based on the truncated representation, as shown to be Consistent, Asymptotically Gaussian and Efficient for $m = T^\beta$ with $\beta \geq 1/2$; see Chan and Palma (1998). Here we adopt the truncated AR(m) representation with $m = \sqrt{T}$. Note that the initialization of $\mathrm{P}_{1|0}$ requires to invert a $m^2 \times m^2$ matrix, thus to reduce the computational complexity we set $\mathrm{P}_{1|0}$ equal to the Toeplitz matrix of $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$, where γ_j are the autocovariances of the long memory process.

A.3 The Kalman filter with level shifts

The standard Kalman filter algorithm can be modified to account for the presence of a level shift process. Following Kim (1994), we show how to obtain the recursive formulas to calculate the transition probabilities and the log-likelihood function for model (7) presented in Section 2.3. The predictive filter transition probability in (11) is obtained as follows

$$\begin{aligned}
\pi^{(i,j)}_{t|t-1} &= \Pr(\mathrm{S}_{t-1} = i, \mathrm{S}_t = j | \mathrm{Y}_{t-1})\\
&= \Pr(\mathrm{S}_t = j | S_{t-1} = i) \Pr(\mathrm{S}_{t-1} = i | \mathrm{Y}_{t-1})\\
&= \Pr(\mathrm{S}_t = j) \Pr(\mathrm{S}_{t-1} = i | \mathrm{Y}_{t-1})\\
&= \lambda_j \pi^{(i)}_{t-1|t-1}.
\end{aligned}$$

Given our model with two state and one transition probability we can express the four filtering probabilities in compact form

$$\mathrm{vec}(\Pi_{t|t-1}) = \begin{bmatrix} \pi & \pi & 0 & 0 \\ 0 & 0 & \pi & \pi \\ 1-\pi & 1-\pi & 0 & 0 \\ 0 & 0 & 1-\pi & 1-\pi \end{bmatrix} \mathrm{vec}(\Pi_{t-1|t-1}), \tag{31}$$

where Π_t is 2×2 matrix

$$\Pi_t = \begin{bmatrix} \pi^{(1,1)}_t & \pi^{(1,2)}_t \\ \pi^{(2,1)}_t & \pi^{(2,2)}_t \end{bmatrix}. \tag{32}$$

The conditional probability for the observation in expression (12) is obtained as follows

$$\begin{aligned}\Pr(y_t|\mathrm{Y}_{t-1}) &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{2}\Pr(y_t|\mathrm{S}_{t-1}=i,\mathrm{S}_t=j,\mathrm{Y}_{t-1})\Pr(\mathrm{S}_{t-1}=i,\mathrm{S}_t=j|\mathrm{Y}_{t-1}) \\ &= \textstyle\sum_{i=1}^{2}\sum_{j=1}^{2}\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)} \\ &= \mathrm{vec}(\Omega_t)'\mathrm{vec}(\Pi_{t|t-1}),\end{aligned} \quad (33)$$

where Ω_t in 2×2 matrix containing the observations' conditional probabilities

$$\Omega_t = \begin{bmatrix} \omega_t^{(1,1)} & \omega_t^{(1,2)} \\ \omega_t^{(2,1)} & \omega_t^{(2,2)} \end{bmatrix}, \quad \omega_t^{(i,j)} = \Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1}). \quad (34)$$

The expression (14) is obtained as follows:

$$\begin{aligned}\pi_{t|t}^{(i,j)} &= \Pr(\mathrm{S}_{t-1}=i,\mathrm{S}_t=j|\mathrm{Y}_t) \\ &= \Pr(\mathrm{S}_{t-1}=i,\mathrm{S}_t=j|y_t,\mathrm{Y}_{t-1}) \\ &= \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\Pr(\mathrm{S}_{t-1}=i,\mathrm{S}_t=j|\mathrm{Y}_{t-1})}{\Pr(y_t|\mathrm{Y}_{t-1})} \\ &= \frac{\Pr(y_t^{(i,j)}|\mathrm{Y}_{t-1})\pi_{t|t-1}^{(i,j)}}{\Pr(y_t|\mathrm{Y}_{t-1})},\end{aligned}$$

this can be express in compact form

$$\mathrm{vec}(\Pi_{t|t}) = \frac{\mathrm{vec}(\Omega_t)\odot\mathrm{vec}(\Pi_{t|t-1})}{\mathrm{vec}(\Omega_t)'\mathrm{vec}(\Pi_{t|t-1})}, \quad (35)$$

where "$\odot$" is the Hadamard product.

B Tables

T	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_{SSF}$
				ARFIMA(0,0.4,0):					
500	0.078	0.013	0.016	0.054	0.077	0.021	0.033	0.080	*0.384*
1000	0.062	0.017	0.021	0.060	0.055	0.029	0.034	0.063	*0.390*
2000	0.060	0.019	0.024	0.061	0.045	0.033	0.036	0.064	*0.392*
				ARFIMA(1,0.4,0), with $\phi = 0.5$:					
500	0.095	0.004	0.004	0.056	0.108	0.010	0.030	0.0760	*0.375*
1000	0.085	0.010	0.011	0.078	0.061	0.015	0.036	0.068	*0.385*
2000	0.067	0.017	0.024	0.064	0.044	0.022	0.046	0.063	*0.394*
				ARFIMA(1,0.4,0), with $\phi = 0.8$:					
500	0.036	0.006	0.015	0.061	0.087	0.071	0.000	0.112	*0.450*
1000	0.046	0.010	0.016	0.059	0.058	0.084	0.000	0.130	*0.447*
2000	0.034	0.002	0.004	0.049	0.045	0.082	0.000	0.105	*0.423*
				ARFIMA(0,0.4,1), with $\theta = 0.5$:					
500	0.065	0.014	0.014	0.060	0.071	0.025	0.034	0.087	*0.370*
1000	0.055	0.020	0.027	0.060	0.047	0.035	0.033	0.066	*0.379*
2000	0.053	0.023	0.032	0.057	0.045	0.034	0.033	0.065	*0.381*
				ARFIMA(0,0.4,1), with $\theta = 0.8$:					
500	0.094	0.017	0.024	0.062	0.060	0.031	0.023	0.075	*0.370*
1000	0.060	0.017	0.026	0.061	0.056	0.033	0.030	0.066	*0.381*
2000	0.054	0.023	0.030	0.059	0.045	0.034	0.033	0.063	*0.385*

Table 1: Empirical Size. The table reports the empirical rejection rate of several test statistics when $\mu_t = 0$ and x_t is generated according to different ARFIMA specifications with $d = 0.4$. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. Finally, last column reports the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2.

T	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_{SSF}$	VR
					i.i.d. Gaussian Noise:					
500	**0.832**	0.183	0.151	0.097	0.183	0.328	0.004	0.128	*-0.028*	0.402
1000	**0.904**	0.264	0.212	0.183	0.297	0.454	0.002	0.184	*0.002*	0.393
2000	**0.967**	0.571	0.374	0.223	0.422	0.669	0.001	0.284	*-0.000*	0.408
					ARFIMA(1,0.2,0), with $\phi = 0.5$:					
500	**0.586**	0.082	0.067	0.067	0.144	0.239	0.005	0.089	*0.174*	0.287
1000	**0.638**	0.193	0.151	0.122	0.169	0.350	0.003	0.110	*0.165*	0.274
2000	**0.739**	0.423	0.274	0.128	0.170	0.550	0.002	0.132	*0.171*	0.283
					ARFIMA(1,0.2,0), with $\phi = 0.8$:					
500	**0.349**	0.011	0.017	0.042	0.112	0.074	0.015	0.047	*0.279*	0.141
1000	**0.348**	0.073	0.010	0.080	0.081	0.131	0.008	0.048	*0.236*	0.128
2000	**0.471**	0.010	0.000	0.087	0.078	0.196	0.004	0.075	*0.208*	0.132
					ARFIMA(0,0.2,1), with $\theta = 0.5$:					
500	**0.628**	0.212	0.206	0.068	0.258	0.269	0.001	0.095	*0.193*	0.319
1000	**0.702**	0.449	0.433	0.141	0.329	0.451	0.003	0.124	*0.194*	0.307
2000	**0.814**	0.692	0.667	0.118	0.495	0.624	0.002	0.135	*0.195*	0.319
					ARFIMA(0,0.2,1), with $\theta = 0.8$:					
500	**0.558**	0.228	0.207	0.073	0.246	0.265	0.003	0.081	*0.196*	0.267
1000	**0.635**	0.431	0.423	0.110	0.349	0.438	0.002	0.102	*0.196*	0.255
2000	**0.752**	0.683	0.652	0.133	0.492	0.641	0.004	0.163	*0.198*	0.266

Table 2: Power. Non-stationary random level shift model. The table reports the empirical rejection rate of several test statistics when μ_t is a random level shift process and x_t is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). *ORT* is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

T	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_{SSF}$	VR
					i.i.d. Gaussian Noise:					
500	**0.554**	0.183	0.162	0.063	0.274	0.288	0.018	0.110	*-0.007*	0.138
1000	**0.807**	0.392	0.353	0.156	0.532	0.525	0.007	0.168	*-0.004*	0.225
2000	**0.970**	0.831	0.764	0.154	0.781	0.660	0.001	0.176	*-0.004*	0.333
					ARFIMA(1,0.2,0), with $\phi = 0.5$:					
500	**0.251**	0.083	0.086	0.064	0.157	0.188	0.012	0.078	*0.189*	0.079
1000	**0.414**	0.280	0.272	0.122	0.258	0.394	0.003	0.106	*0.195*	0.131
2000	0.623	**0.675**	0.653	0.144	0.465	0.639	0.002	0.167	*0.201*	0.200
					ARFIMA(1,0.2,0), with $\phi = 0.8$:					
500	**0.263**	0.005	0.002	0.064	0.100	0.036	0.027	0.053	*0.247*	0.036
1000	**0.365**	0.008	0.012	0.107	0.083	0.082	0.010	0.055	*0.227*	0.059
2000	**0.464**	0.007	0.004	0.090	0.072	0.183	0.004	0.066	*0.211*	0.092
					ARFIMA(0,0.2,1), with $\theta = 0.5$:					
500	**0.268**	0.086	0.088	0.059	0.154	0.177	0.016	0.073	*0.207*	0.100
1000	**0.492**	0.275	0.265	0.130	0.237	0.377	0.006	0.105	*0.193*	0.164
2000	**0.703**	0.684	0.643	0.133	0.463	0.630	0.001	0.153	*0.200*	0.247
					ARFIMA(0,0.2,1), with $\theta = 0.8$:					
500	**0.251**	0.083	0.086	0.064	0.157	0.188	0.012	0.078	*0.189*	0.079
1000	**0.414**	0.280	0.272	0.122	0.258	0.394	0.003	0.106	*0.195*	0.131
2000	0.623	**0.675**	0.653	0.144	0.465	0.639	0.002	0.167	*0.201*	0.200

Table 3: Power - Stationary random level shift model. The table reports the empirical rejection rate of several test statistics when μ_t is a stationary random level shift process and x_t is generated according to different ARFIMA specifications. Results are based on $M = 1000$ Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

T	SSF$_k$	QU$_{2\%}$	QU$_{5\%}$	ORT	PQ	SH$_k$	SH$_p$	SH$_s$	$\hat{d}_{SSF}$	VR
				i.i.d. Gaussian Noise:						
500	**0.942**	0.013	0.015	0.040	0.222	0.190	0.001	0.050	*0.001*	0.113
1000	**1.000**	0.010	0.013	0.058	0.498	0.210	0.001	0.032	*-0.004*	0.116
2000	**1.000**	0.020	0.016	0.072	0.729	0.515	0.001	0.050	*-0.004*	0.119
				ARFIMA(1,0.2,0), with $\phi = 0.5$:						
500	**0.757**	0.006	0.012	0.069	0.101	0.089	0.006	0.067	*0.200*	0.060
1000	**0.929**	0.020	0.026	0.078	0.093	0.146	0.005	0.058	*0.174*	0.062
2000	**0.987**	0.046	0.042	0.072	0.074	0.258	0.002	0.047	*0.159*	0.065
				ARFIMA(1,0.2,0), with $\phi = 0.8$:						
500	**0.948**	0.015	0.018	0.058	0.206	0.173	0.001	0.049	*0.194*	0.063
1000	**0.996**	0.017	0.021	0.082	0.225	0.291	0.001	0.060	*0.195*	0.065
2000	**1.000**	0.045	0.036	0.065	0.271	0.449	0.001	0.055	*0.199*	0.068
				ARFIMA(0,0.2,1), with $\theta = 0.5$:						
500	**0.989**	0.013	0.016	0.055	0.187	0.172	0.001	0.043	*0.193*	0.074
1000	**1.000**	0.013	0.015	0.090	0.197	0.257	0.001	0.049	*0.198*	0.075
2000	**1.000**	0.050	0.038	0.098	0.275	0.448	0.001	0.066	*0.199*	0.077
				ARFIMA(0,0.2,1), with $\theta = 0.8$:						
500	**0.948**	0.015	0.018	0.058	0.206	0.173	0.001	0.049	*0.194*	0.063
1000	**0.996**	0.017	0.021	0.082	0.225	0.291	0.001	0.060	*0.195*	0.065
2000	**1.000**	0.045	0.036	0.065	0.271	0.449	0.001	0.055	*0.199*	0.068

Table 4: Power - Monotonic trend model. The table reports the empirical rejection rate of several test statistics when μ_t is a monotonic trend and x_t is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF$_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH$_k$ denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH$_p$ is its PhillipsPerron version. SH$_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

T	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_{SSF}$	VR
				i.i.d. Gaussian Noise:						
500	**0.993**	0.839	0.491	0.079	0.085	0.003	0.001	0.099	*-0.035*	0.334
1000	**1.000**	0.954	0.608	0.141	0.075	0.005	0.001	0.099	*-0.025*	0.334
2000	**1.000**	**1.000**	0.722	0.162	0.095	0.015	0.001	0.124	*-0.017*	0.335
				ARFIMA(1,0.2,0), with $\phi = 0.5$:						
500	0.070	**0.126**	0.072	0.079	0.106	0.016	0.003	0.066	*0.319*	0.208
1000	0.235	**0.286**	0.162	0.109	0.096	0.030	0.001	0.081	*0.267*	0.203
2000	**0.661**	0.602	0.337	0.103	0.064	0.082	0.001	0.099	*0.157*	0.202
				ARFIMA(1,0.2,0), with $\phi = 0.8$:						
500	**0.131**	0.008	0.008	0.062	0.110	0.022	0.016	0.066	*0.265*	0.083
1000	**0.150**	0.011	0.010	0.091	0.094	0.019	0.009	0.044	*0.247*	0.079
2000	**0.211**	0.006	0.006	0.061	0.084	0.033	0.005	0.041	*0.220*	0.075
				ARFIMA(0,0.2,1), with $\theta = 0.5$:						
500	0.243	**0.449**	0.286	0.075	0.074	0.002	0.001	0.067	*0.257*	0.241
1000	0.501	**0.660**	0.589	0.115	0.079	0.020	0.001	0.107	*0.210*	0.237
2000	0.887	**0.962**	0.921	0.131	0.040	0.186	0.001	0.133	*0.205*	0.237
				ARFIMA(0,0.2,1), with $\theta = 0.8$:						
500	**0.131**	0.008	0.008	0.062	0.110	0.022	0.016	0.066	*0.265*	0.083
1000	**0.150**	0.011	0.010	0.091	0.094	0.019	0.009	0.044	*0.247*	0.079
2000	**0.211**	0.006	0.006	0.061	0.084	0.033	0.005	0.041	*0.220*	0.075

Table 5: Power - Nonmonotonic trend model. The table reports the empirical rejection rate of several test statistics when μ_t is a non-monotonic trend and x_t is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

T	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_{SSF}$	VR
					i.i.d. Gaussian Noise:					
500	**0.452**	0.201	0.203	0.062	0.347	0.260	0.021	0.079	*-0.008*	0.545
1000	**0.669**	0.415	0.392	0.108	0.573	0.341	0.017	0.110	*-0.007*	0.578
2000	0.845	0.688	0.659	0.153	**0.847**	0.453	0.011	0.145	*-0.007*	0.613
					ARFIMA(1,0.2,0), with $\phi = 0.5$:					
500	**0.375**	0.028	0.026	0.054	0.122	0.095	0.012	0.078	*0.278*	0.388
1000	**0.538**	0.079	0.061	0.117	0.142	0.262	0.011	0.102	*0.254*	0.414
2000	**0.773**	0.261	0.160	0.115	0.158	0.448	0.004	0.127	*0.223*	0.441
					ARFIMA(1,0.2,0), with $\phi = 0.8$:					
500	0.029	0.004	0.005	0.057	0.104	**0.040**	0.024	0.053	*0.437*	0.179
1000	0.027	0.006	0.007	0.076	0.076	**0.104**	0.006	0.057	*0.452*	0.195
2000	0.010	0.009	0.008	0.072	0.068	**0.215**	0.002	0.054	*0.460*	0.212
					ARFIMA(0,0.2,1), with $\theta = 0.5$:					
500	**0.426**	0.099	0.087	0.053	0.234	0.208	0.021	0.101	*0.198*	0.429
1000	**0.609**	0.342	0.306	0.107	0.488	0.348	0.015	0.094	*0.200*	0.464
2000	**0.845**	0.683	0.612	0.164	0.765	0.509	0.009	0.176	*0.201*	0.497
					ARFIMA(0,0.2,1), with $\theta = 0.8$:					
500	**0.408**	0.123	0.107	0.062	0.300	0.187	0.020	0.070	*0.191*	0.362
1000	**0.593**	0.364	0.346	0.112	0.571	0.346	0.015	0.092	*0.198*	0.359
2000	0.819	0.715	0.692	0.149	**0.847**	0.506	0.007	0.157	*0.205*	0.416

Table 6: Power - Markov switching model. The table reports the empirical rejection rate of several test statistics when μ_t is a Markov-switching and x_t is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

T	SSF$_k$	QU$_{2\%}$	QU$_{5\%}$	ORT	PQ	SH$_k$	SH$_p$	SH$_s$	$\hat{d}_{SSF}$	VR
					i.i.d. Gaussian Noise:					
500	**0.139**	0.021	0.027	0.048	0.122	0.066	0.079	0.078	*0.141*	0.900
1000	0.249	0.055	0.042	0.132	0.122	**0.255**	0.057	0.086	*0.142*	0.857
2000	0.441	0.198	0.093	0.128	0.128	**0.480**	0.022	0.155	*0.146*	0.845
					ARFIMA(1,0.2,0), with $\phi = 0.5$:					
500	**0.111**	0.005	0.013	0.043	0.112	0.026	0.034	0.068	*0.102*	0.762
1000	**0.233**	0.015	0.016	0.087	0.071	0.114	0.018	0.062	*0.122*	0.0.759
2000	**0.368**	0.036	0.022	0.098	0.080	0.244	0.010	0.072	*0.138*	0.742
					ARFIMA(1,0.2,0), with $\phi = 0.8$:					
500	0.005	0.001	0.005	0.063	0.097	0.009	0.022	0.052	*0.298*	0.473
1000	0.009	0.004	0.004	**0.088**	0.081	0.062	0.008	0.059	*0.301*	0.485
2000	0.009	0.019	0.016	0.063	0.076	**0.118**	0.007	0.028	*0.309*	0.495
					ARFIMA(0,0.2,1), with $\theta = 0.5$:					
500	**0.112**	0.002	0.004	0.052	0.119	0.073	0.064	0.071	*0.094*	0.813
1000	**0.307**	0.055	0.058	0.114	0.106	0.258	0.057	0.085	*0.096*	0.801
2000	0.484	0.189	0.109	0.135	0.123	**0.486**	0.027	0.134	*0.104*	0.781
					ARFIMA(0,0.2,1), with $\theta = 0.8$:					
500	**0.159**	0.024	0.036	0.062	0.135	0.077	0.082	0.074	*0.080*	0.731
1000	**0.297**	0.068	0.060	0.128	0.117	0.244	0.058	0.089	*0.088*	0.730
2000	0.505	0.262	0.153	0.149	0.154	**0.527**	0.029	0.168	*0.093*	0.729

Table 7: Power - Markov switching model with GARCH errors. The table reports the empirical rejection rate of several test statistics when μ_t is a Markov-switching term with GARCH errors and x_t is generated according to different ARFIMA specifications. Results are based on 1000 Monte Carlo replications. For each test, the null is true fractional integration (i.e. absence of shifts). SSF$_k$ is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). *PQ* is the test of Perron and Qu (2010). SH$_k$ denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH$_p$ is its PhillipsPerron version. SH$_s$ is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_{SSF}$ is the Monte Carlo average of the estimates of the d parameter based on the state-space methodology outlined in Section 2. Finally, last column reports the Monte Carlo average of the variance ratio, VR, between the sample variance of μ_t and to the sample variance of y_t.

	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_w$	$\hat{d}_{SSF}$
BA	0.638*	0.868	0.558	3.035	-0.298	-1.606	0.118	2.881	0.652	0.420
BAC	1.735*	0.653	0.569	7.415	2.493*	-0.802	0.226	3.169	0.711	0.412
C	2.710*	0.421	0.685	7.224	2.589*	-0.948	0.265	2.064	0.702	0.395
CAT	0.318	1.080	0.721	1.286	0.191	-1.656	0.114	7.307	0.707	0.477
FDX	0.732*	0.601	0.539	1.331	1.177	-1.166	0.195	4.251	0.617	0.373
HON	0.355	0.771	0.460	1.099	-0.360	-1.195	0.097	9.070*	0.645	0.404
HPQ	0.568*	0.425	0.540	0.681	-0.455	-2.203	0.078	3.474	0.668	0.339
IBM	0.283	0.488	0.845	2.184	0.454	-2.334	0.058	6.114	0.705	0.447
JPM	1.419*	0.517	0.633	7.361	1.810	-1.607	0.212	3.845	0.716	0.412
PEP	0.426	0.365	0.468	4.374	-0.270	-1.664	0.084	6.159	0.699	0.436
PG	0.166	0.565	0.670	4.548	0.749	-2.098	0.058	8.335*	0.673	0.499
T	0.313	0.449	0.670	2.390	-0.410	-0.931	0.087	4.326	0.680	0.429
TWX	0.584*	0.449	0.553	1.539	-0.079	-1.508	0.080	9.324*	0.702	0.450
TXN	0.517*	0.755	0.314	1.072	-0.492	-1.382	0.123	5.775	0.695	0.442
WFC	2.492*	0.594	0.368	4.773	2.029*	-1.110	0.234	2.174	0.740	0.371

Table 8: Empirical application. The table reports the values of several test statistics for the bipower-variations series of 15 assets traded on NYSE. The asterisk denotes rejection of the null at 5% significance level. SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_w$ and $\hat{d}_{SSF}$ are the estimates of the fractional parameter obtained with the local Whittle estimator and the state-space method respectively.

	SSF_k	$QU_{2\%}$	$QU_{5\%}$	ORT	PQ	SH_k	SH_p	SH_s	$\hat{d}_w$	$\hat{d}_{SSF}$
BA	0.767*	2.382*	1.926*	20.43*	1.859	-0.356	0.794*	0.753	0.394	0.275
BAC	9.357*	0.918	0.749	5.359	1.673	-0.066	0.563*	5.351	0.831	0.262
C	5.156*	1.699*	1.248*	6.271	1.268	-0.444	0.291	19.14*	0.630	0.304
CAT	0.272	1.581*	1.283*	3.594	1.717	-0.635	0.477*	12.55*	0.446	0.385
FDX	1.255*	1.753*	1.293*	0.066	0.561	-0.507	0.665*	3.411	0.357	0.246
HON	0.523*	1.025	0.641	9.005*	1.879	-0.898	0.362	13.30*	0.410	0.346
HPQ	6.707*	2.283*	1.884*	4.282	2.087*	1.060	1.779*	15.43*	0.425	0.213
IBM	0.987*	1.015	0.588	3.514	2.084*	-1.024	0.261	2.333	0.431	0.286
JPM	7.063*	1.857*	1.450*	10.81*	2.753*	-0.682	0.516*	7.295	0.546	0.261
PEP	0.735*	1.892*	1.201*	3.507	2.706*	-0.345	0.803*	0.918	0.427	0.357
PG	0.393	0.871	0.703	1.598	0.714	-0.736	0.372	20.84*	0.444	0.375
T	0.512*	0.922	0.499	2.873	1.641	-0.905	0.346	6.204	0.374	0.319
TWX	0.510*	1.164	1.095	2.954	1.794	-0.390	0.564*	21.27*	0.397	0.335
TXN	0.609*	1.392*	0.768	1.896	1.439	-0.525	0.528*	5.053	0.454	0.331
WFC	2.890*	1.061	0.497	5.978	2.064*	-0.641	0.410	11.52*	0.723	0.311

Table 9: Empirical application. The table reports the values of several test statistics for the daily turnover series of 15 assets traded on NYSE. The asterisk denotes rejection of the null at 5% significance level. SSF_k is the KPSS test based on the state-space representation. QU denotes the Qu (2011) test based on the local Whittle likelihood, with two different trimming choices ($\epsilon = 2\%$ and $\epsilon = 5\%$). ORT is the temporal aggregation test of Ohanissian et al. (2008). PQ is the test of Perron and Qu (2010). SH_k denotes the KPSS test of Shimotsu (2006) based on d^{th}-differencing and SH_p is its PhillipsPerron version. SH_s is the Shimotsu (2006) test based on sample splitting with 4 sub-samples. $\hat{d}_w$ and $\hat{d}_{SSF}$ are the estimates of the fractional parameter obtained with the local Whittle estimator and the state-space method respectively.

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