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Appropriate Technology and the Labour Share

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Please see instead the updated 2016 version of this working paper in the University of Kent School of Economics Discussion Papers series KDPE 1614 under the title
“Appropriate Technology and Balanced Growth.”

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Abstract

Please see instead the updated 2016 version of this working paper in the University of Kent School of Economics Discussion Papers series under the title "Appropriate Technology and Balanced Growth." We provide a general theoretical characterization of how technology choice affects the long-run elasticity of substitution between capital and labour. While the shape of the technology frontier determines the long-run growth path and the long-run elasticity, adjustment costs in technology choice allow capital-labour complementarity in the short run. We develop a class of production functions that are consistent with balanced growth even in the presence of permanent investment-specific or other kinds of biased technical progress but where, consistent with empirical evidence, short-run dynamics are characterized by complementarity. Importantly, the approach is easily implementable and yields a powerful way to introduce CES-type production functions in macroeconomic models. We provide an illustration within an estimated dynamic general equilibrium model and show that the use of the new production technology provides a good match for the short and medium run behavior of the US labour share.

JEL Classification: E25, O33, O40.
Keywords: Balanced growth, appropriate technology, elasticity of substitution.

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Non-technical summary

In macroeconomics, we typically model production by specifying a 'production function,' which tells us how much output is produced with given quantities of the 'factors of production,' often taken simply as capital and labour. Factor shares refer to the proportion of the income earned by production that goes to each factor, so the labour share is the proportion of this income that is earned by workers through supplying labour. There are various issues with how we measure factor shares, but a key aspect of what is known as 'balanced growth' is the idea that as income grows over long periods of time, the labour share remains approximately constant.

Some researchers dispute the idea of balanced growth, arguing for example that the labour share is currently declining. What there is no disagreement about is the fact that these factor shares are far more stable in the long run than they are in the short run. This creates a problem in the way we specify production functions. For example, the assumption of balanced growth has led Cobb-Douglas production functions to become standard in macroeconomic models because they imply constant factor shares with perfectly competitive markets. This, however, makes it more difficult to capture short- and medium-run fluctuations in factor shares. Market failures such as wage and price rigidities allow us to explain some of these fluctuations, but it is unlikely that they account for all of the fluctuations we see in factor shares, particularly in the medium run.

When the relative price of capital (to labour) rises, firms hire relatively less capital and more labour. The elasticity of substitution between capital and labour quantifies this effect; it tells us by how many percent the capital-labour ratio declines when the relative price of capital goes up by 1%. With Cobb-Douglas production functions, this elasticity is always one. However, much of the empirical evidence finds support for an elasticity below one. Indeed, production functions with an elasticity below 1 typically capture short-run fluctuations in factor shares significantly better than Cobb-Douglas. However, they have very important long-run consequences for income distribution. If the elasticity is different to one, productivity changes can cause the labour share to change. Since we have observed permanent changes in the productivity of investment goods in the last 30 years, an elasticity below one would lead to unbalanced growth with an increasing labour share, whereas typically researchers think that it is either constant or declining.

In this paper we propose a solution to this problem, using the idea of "appropriate technology." This is the idea that firms not only choose the quantities of capital and labour to employ, but also make a technology choice – how labour- or capital-intensive they want production methods to be. This trade-off is expressed graphically by a technology frontier: technologies that are more efficient in using labour are less efficient in using capital and vice-versa. Given a change in factor prices, firms change their position on the frontier. We show how the shape of the frontier determines the long-run elasticity of substitution and long-run factor shares. Importantly, if firms face adjustment costs when changing their choice of technology, the short-run elasticity will be lower than the long-run elasticity. This provides a way of modelling production that is very easy to implement in macroeconomic models but that is flexible enough to be compatible with both short- and long-run data. The short-run elasticity can be calibrated to capture short-run fluctuations in factor shares in line with the evidence, while the shape of the frontier captures the properties of long-run growth. There is a specific shape of frontier that implies balanced growth. Here elasticity of substitution is below one in the short-run but adjusts towards one in the long run. We use this to provide a quantitative example for the US economy. The results support the use of this new production function because it improves the model's ability to explain the business cycle and medium-run behaviour of the labour share.

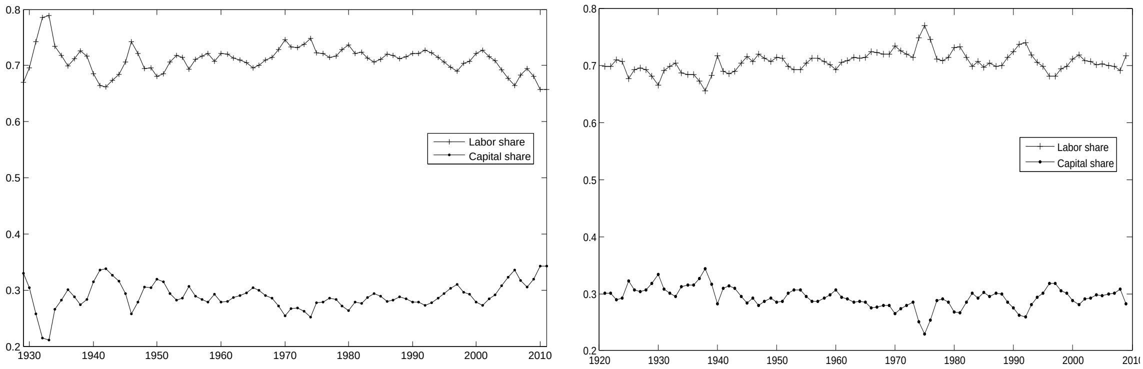
1 Introduction

The Steady State Growth Theorem (Uzawa, 1961; henceforth *SSGT*) constitutes a significant constraint on macroeconomic modelling. It states that balanced growth requires all technical progress to be labour-augmenting or the elasticity of substitution between capital and labour to equal one in the long run (see Jones and Scrimgeour, 2008, for a useful proof). Evidence that factor shares are approximately constant in the long run, such as shown in figure 1, has led to balanced growth being a standard baseline description of long-run data and also a standard constraint for most solution methods in dynamic macro models. However, as discussed below, the assumption that technical progress is indeed *purely* labour augmenting is difficult to justify. On the other hand, Cobb-Douglas, which imposes a unitary capital-labour elasticity of substitution, sits at odds both with the substantial *cyclical* fluctuations observed in factor shares and the weight of evidence (reviewed in e.g. Chirinko, 2008, and León-Ledesma *et al.*, 2010) which supports a value of this elasticity significantly below unity at standard frequencies. Specifying an elasticity less than unity might be beneficial in modelling fluctuations, but doing so precludes long-run balanced growth unless all technical progress is labour augmenting. It is in this sense that the theorem constrains modelling practice.

We propose a method for relaxing this constraint by deriving a general production function in which $\sigma < 1$ and $\sigma < \sigma_{LR}$ where σ and σ_{LR} are respectively the short- and long-run values of the elasticity of substitution between capital and labour. The special case where $\sigma_{LR} = 1$ ensures balanced growth. This allows flexibility in modelling short-run dynamics, while retaining compatibility with balanced growth without imposing any restrictions on the nature of technical progress or investment-specific technical change. A particular focus is to provide a tractable method of achieving this flexibility. In comparison to alternative approaches, this method can be more easily applied in a wide range of macroeconomic models commonly used for policy making, where the nature of technical progress is not the primary research question.

The dynamic behavior of the elasticity of substitution between capital and labour (henceforth *ES*) in this production function arises through a choice of (appropriate) technology by the firm. This type of choice is analysed in the appropriate technology literature (see Atkinson and Stiglitz, 1969, Caselli and Coleman, 2006 and Jones, 2005) which forms the basis for the current paper. On a country level, the central idea is that countries with different factor endowments choose technologies that are appropriate according to their factor abundance. The paper provides a general theoretical characterization of how the shape of the relevant technology frontier affects the relationship between σ and σ_{LR}, and shows that balanced growth arises when this choice takes a particularly simple form.

Here, we build on the approach of Caselli and Coleman (2006). Firms choose the efficiencies of capital and labour where these efficiencies are constrained by a technology frontier. Since the choice of technology will depend on factor prices, and the quantities of factors employed depend on the choice of technology, technology choice influences the *ES*. One straightforward contribution of the paper is to propose that a tractable method of obtaining time variation in the *ES* is to impose (deterministic) adjustment costs on the choice of technology (see Jones, 2005, and the discussion below). Following a shock that changes factor prices, the firm's response in the relative quantities of factors employed will be different 'before' and 'after' adjustment of technology has occurred. It is the corresponding 'before' and 'after' values of the *ES* that we denote σ and σ_{LR} respectively. The theoretical framework characterizes conditions for an interior solution under technology choice for a general frontier, and how the shape of the frontier affects the relationship between σ and σ_{LR}. Setting $\sigma_{LR} = 1$ for compatibility with balanced growth, we then present an illustrative dynamic macro model that, despite its simplicity, captures well the short- and medium-run behavior of the labour share.



(a) US 1929-2011 (b) UK 1921-2012

Figure 1: Historical capital and labour shares for the US and UK. Sources: US, Piketty and Saez (2003), UK, Mitchell (1988) updated by Bank of England.

Empirical Context. It is useful to discuss the empirical context of the *SSGT* in more detail, starting with the assumption of long-run balanced growth itself. Some recent literature (see e.g. Piketty, 2013 and Karabarbounis and Nieman, 2014) has argued that current trends suggest that the capital share is increasing over time, and so growth is not balanced but best described by $\sigma_{LR} > 1$. This argument is far from settled, since observations on the capital share and the capital-output ratio are disputed due to measurement issues (see e.g. Elsby et al., 2013, Bonnet et al, 2014, and Bridgman, 2014).[1] The essential point for this exercise, however, is to note that since short-run evidence favors $\sigma < 1$,[2] either of these two views of long-run growth

[1]Note also theoretical arguments such as those of La Grandville (2012), who shows that the combination of technical progress and $\sigma_{LR} > 1$ leads to a positively trended and unbounded real interest rate.

[2]See, again, Chirinko (2008), Klump et al. (2007), León-Ledesma et al. (2010, 2014).

suggest $\sigma_{LR} > \sigma$. We argue that modelling technological choice is a good way of achieving this.

Suppose then we take compatibility with long-run balanced growth as a model requirement. The standard method is to assume a Cobb-Douglas production function, where $\sigma = \sigma_{LR} = 1$. As is well known, under perfect competition or constant firm markups, this implies constant factor shares in both the long and the short run. The latter is clearly counterfactual, but this problem can potentially be addressed by introducing wage and price rigidities, which may generate short-run variations in factor shares if they produce cyclical fluctuations in firm markups. While such rigidities are likely to affect the labour share at business cycle frequencies, at medium-run frequencies we might expect wages and prices to adjust. Thus, they cannot be the driver of observed medium-run fluctuations in factor shares. If $\sigma < 1$, however, technical change might play a role in explaining such medium-run fluctuations.[3] Furthermore, estimates of capital income built by directly calculating the real user cost as in Klump et al. (2007) display considerable business cycle fluctuations that cannot be attributed to changing markups. Bentolila and Saint-Paul (2003) also find evidence that changes in the labour share are significantly driven by technological shifts unrelated to labour market rigidities.

As discussed above, the primary alternative offered by the $SSGT$ is the assumption that technical progress is purely labour-augmenting in the long run. Suppose for argument's sake we assume a CES (Constant Elasticity of Substitution) production function (where $\sigma = \sigma_{LR}$). If $\sigma = \sigma_{LR} \neq 1$, technology shocks will produce short-run fluctuations in factor shares (with typically $\sigma = \sigma_{LR} < 1$), potentially allowing models to capture these fluctuations while satisfying balanced growth. Some potential disadvantages with such an approach are as follows. Firstly, it is difficult to make a clear theoretical case as to why any permanent technical progress should be purely labour augmenting (discussed further below).[4] If one is unwilling to make the unattractive assumption that temporary technical progress is an important driver of cyclical fluctuations, the limitation to purely labour-augmenting technical progress may also fail to capture some aspects of short-run dynamics. Here, permanent capital- and labour-augmentation are distinguished in terms of their short-run dynamics (not in the long-run, where we have Cobb-Douglas). So the possibility of capital-augmenting technical progress may allow a model to better capture observed short-run dynamics.

Empirically, the important role investment-specific technical change (henceforth IST) has taken in macroeconomics in the past two decades also militates against the view of technical progress as purely labour-augmenting. For example, Greenwood et al. 1997 and 2000, and Fisher, 2006, find IST to be one of the key drivers of

[3] The empirical exercise in the paper distinguishes between short- and medium-run movements, following Comin and Gertler (2006).

[4] Note also that the joint assumptions of a CES production function and purely labour-augmenting technical progress imply long-run cointegration of the log of the capital share and the log of the user cost of capital. See figure 1 in appendix D.

macroeconomic fluctuations in the US economy. IST is clearly not temporary: it seems unlikely theoretically that such technical change is to be reversed and, moreover, relative price data for investment goods, a proxy for IST, are clearly trended. Since IST has similar implications for balanced growth as capital-augmenting technical progress, trends in IST will not result in balanced growth in conventional models with a CES production function. Nonetheless, our approach provides a justification for the use of CES production functions in modelling short-run dynamics, since any such model can potentially be made compatible with balanced growth by the introduction of technological choice.[5]

Related Theory. Theoretical reasons for why technical progress may be purely labour-augmenting are examined in the "induced innovation" strand of the literature. The early literature on induced innovation by Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966) and Kamien and Schwarz (1968), inspired by Hicks (1932), viewed this as the result of firms introducing innovations that save on expensive factors in the face of changes in relative factor prices. More recently, this line of thought has been extended by Acemoglu (2002, 2003, 2007) and Zeira (1998) amongst others. An adequate survey is beyond the scope of this paper, but the question of whether the induced innovation literature as a whole produces the outcome of balanced growth without overly-restrictive assumptions on the nature of innovation is not clear (see Acemoglu, 2003, for a useful discussion). At the very least, to address the problem of balanced growth, the induced innovation literature requires a formal modelling of innovation. This can potentially make departures from the standard Cobb-Douglas framework difficult when the research question does not concern innovation itself or in situations where it is convenient to treat technical progress as exogenous.

This paper is more closely related to a smaller literature that describes models of technological choice, or, equivalently, of 'appropriate technology.' Prominent examples are Jones (2005), in turn extended in various interesting ways by Growiec (2008 and 2013), and Caselli and Coleman (2006). In these approaches, the firm typically makes a technological choice by selecting e.g. a pair (A, B) that represents the efficiency of its two inputs to production. The space of available technologies might take the form of a deterministic frontier in (A, B) space (Caselli and Coleman, 2006), or represent an accumulated stock of arrived individual technologies (A_i, B_i) drawn stochastically from given distributions (Jones, 2005 and Growiec, 2008 and 2013).

In Jones (2005), firms choose the most appropriate of the technologies that have arrived, each of which is Leontief (or CES with a low elasticity of substitution). In the short run, while the firm remains on the current technology, the elasticity of

[5]The introduction of CES production technologies in business cycle analysis has gained relevance in recent years due to an increasing interest in the drivers of factor income shares (see Choi and Ríos-Rull, 2009, and Ríos-Rull and Santaeulàlia-Llopis, 2009). Cantore et. al (2014), for instance, show that the effect of technology shocks on hours worked can solve the technology-hours correlation puzzle when the elasticity of factor substitution differs from one and there are biased technology shocks.

substitution is zero (or low), but in the long run switching to new technologies will cause the elasticity of substitution to increase. Jones (2005) shows that if A and B are drawn from independent Pareto distributions, the elasticity of substitution is unity in the long run. Like our present approach, it produces Cobb-Douglas at the firm level in the long run, rather than as a result of aggregation.[6] Growiec (2013), in turn, shows how the use of Weibull distributions can lead to long-run CES.

While these papers provide a rich and elegant description of technical progress, applying these ideas in a conventional macroeconomic modelling framework is difficult. For instance, due to the fact that firms switch to the best available technology, the dynamics in Jones (2005) have an extreme value property that makes simulation difficult in a conventional forward-looking macroeconomic setting such as a DSGE using the usual solution techniques.[7] Caselli and Coleman (2006, henceforth CC) is not explicitly related to balanced growth. However, as we show here, extending the CC framework to address this and the more general question of creating a production function with a time-varying elasticity of substitution results in a formulation that is very straightforward to introduce into conventional macroeconomic models.

The rest of the paper is organized as follows. The next section contains the key theoretical results. It presents the production technology and its core characteristics, and discusses the firm's problem, dynamics, and relationship to balanced growth. Section 3 presents an application of this approach to modelling the behavior of the labour share of income. Section 4 concludes.

2 The Production Technology

In CC and Jones (2005), firms choose an appropriate technology in light of the relative price of factor inputs. In both of these papers, each technology is represented by a pair of efficiencies for two of the inputs within the same underlying CES production function. We use σ to denote the elasticity of substitution between these two inputs *for a given technology*, writing $\rho \equiv \frac{\sigma-1}{\sigma} < 1$. In Jones (2005) technologies accumulate by a stochastic arrivals process, and firms, having chosen a particular technology, only change when a more appropriate one arrives. In the short run, the inability to change technologies implies that the elasticity of substitution between the two inputs, capital and labour, is σ. The long-run elasticity, denoted σ_{LR}, however, is influenced

[6] The aggregation approach is taken by Houthakker (1955-56). Jones (2005) and Lagos (2006) provide useful discussions of this classic paper. Lagos (2006), in the spirit of Houthakker, derives a Cobb-Douglas form for the aggregate production function by aggregating Leontief production technologies at the firm level using a model with search frictions (assuming an exogenous rental on capital). Since we principally aim at providing a production function, the aims are very different from those here and are primarily directed at accounting for the determinants of observed TFP.

[7] The extreme value property will also have a significant impact on the short-run dynamics of factor shares, increasing the likelihood of sharp adjustments. Here, a standard adjustment cost mechanism results in smoother changes in factor shares.

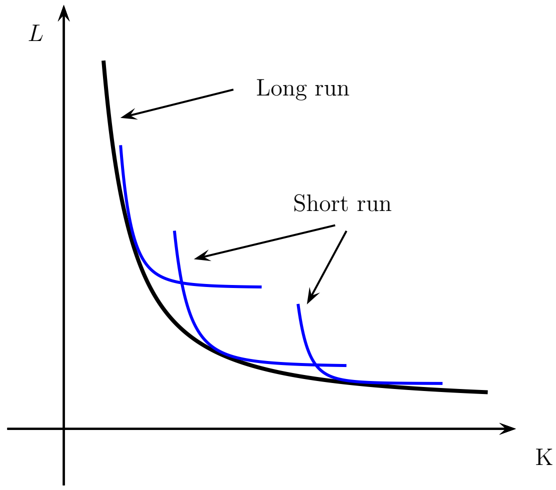


Figure 2: Isoquants of the long- and short-run production functions

by technology choice. As shown in Jones (2005), economic intuition strongly suggests that $\sigma_{LR} > \sigma$. In figure 2, which is very similar to figure 1 of Jones (2005), the isoquants of the 'global' production function that incorporates endogenous technology choice are the convex hull of the isoquants of the individual technologies, and therefore the former should have less curvature than the latter.

On the technology frontier – the set of technologies which are not dominated by another – choosing a technology that makes one input more efficient must, by definition, cause the other to be less efficient. In CC, whose approach we follow here, this frontier is continuous and deterministic (or to be precise it depends continuously and deterministically on at most a small number of stochastic parameters), and represents the firm's choice of the relative efficiencies of skilled and unskilled labour. CC use this idea to derive and estimate a 'world technology frontier' in these inputs, the position of each country's frontier depending on which one of three income groups it falls into. This has three consequences which are important in the current context: (i) it is hard to argue that balanced growth applies empirically to the inputs of skilled and unskilled labour (ii) similarly, there is no pressing reason to differentiate between the short- and long-run elasticity of substitution between them and, finally, (iii) data clearly suggest that the natural assumption is $\sigma > 1$ (or $\rho > 0$) for the two inputs of skilled and unskilled labour.

Since our inputs are capital and labour, the converse of all three statements applies. For (ii), as discussed above, we argue that a natural and effective way to create a difference between the short- and long-run elasticities of substitution is to assume that firms face conventional adjustment costs in their choice of technology. The phrase 'long-run' in the paper refers to the situation where complete adjustment has taken place. For (iii), motivated by the evidence for short-run complementarity between capital and labour, the natural assumption is $\rho < 0$. In the theoretical results that follow, we will initially also allow $\rho > 0$ for generality (though the case $\rho = 0$ is always excluded) but the assumption $\rho < 0$ will be imposed from section 2.3.1 onwards.

2.1 The technology frontier

In CC, there are three inputs, capital, unskilled labour and skilled labour, with technological choice only applying to the latter two. Here we will just have capital and labour as in Jones (2005). Output Y is given by

$$Y = [A^\rho K^\rho + B^\rho L^\rho]^{1/\rho} \tag{1}$$

where, given factor prices, in addition to choosing K and L, the firm also chooses $A, B \geq 0$ representing its appropriate choice of technology. In CC, the technology possibilities set takes the following form:

$$A^\omega + \gamma B^\omega \leq X. \tag{2}$$

We assume throughout that $\rho \neq 0$. In order to examine balanced growth and other outcomes, we are interested in exploring different possible shapes of the possibilities set, so we generalise (2) as follows,

$$\Phi_A \left(A\mathrm{e}^{-x}\right) + \Phi_B \left(B\mathrm{e}^{-x}\right) \leq 0 \tag{3}$$

where $\Phi_i(.)$ are strictly increasing and twice continuously differentiable on $[0, \infty)$ for $i = A, B$ and $x = \ln X$ is a shift parameter that represents a Hicks neutral change of the frontier. We could also add a tilt parameter to (3), but such parameters will emerge naturally in the functional forms that describe the frontier. Assuming the firm always chooses a technology on the frontier, it is useful to describe the firm's position on it in terms of a single variable. This will allow us to eliminate a state variable from the model.

The next steps of the paper proceed as follows. We will not use directly the functions $\Phi_i(.)$ to describe the shape of the frontier, or the variables A or B to locate the firm's position on it. The functions and variables that we do use are described in subsection 2.1.1, which also introduces some geometric properties of the frontier that will be important in determining long-run outcomes. As discussed in subsection 2.1.2, the presence of adjustment costs in technology choice means that the short-run implications of the production process are entirely standard. Therefore the bulk of the theory is devoted to describing how the shape of the frontier affects long-run outcomes. Subsection 2.1.3 describes the solution to the firm's long-run problem, providing sufficient conditions for an interior solution. It also shows how the slope and curvature of the frontier determine the capital share and the long-run elasticity of substitution respectively.

2.1.1 The geometry of the frontier

On the frontier, (3) holds with equality. Since $\Phi_A(.)$ and $\Phi_B(.)$ are strictly increasing in A and B, on a suitable support there is a twice continuously differentiable function $\Psi(\theta)$ such that $\Phi_A\big(\Psi(\theta)\big) + \Phi_B\big(\theta\Psi(\theta)\big) = 0$ (given any θ the left-hand side must be strictly increasing in Ψ, and $\Psi(\theta)$ therefore must be strictly decreasing in θ). Hence we can define the unique point where the frontier in (A, B) space intersects the ray $B = \theta A$ as $\big(A(\theta; x), B(\theta; x)\big)$, where then $A(\theta; x) = \mathrm{e}^x\Psi(\theta)$ and $B(\theta; x) = e^x\theta\Psi(\theta)$. Working in log-space will simplify the derivations, so we write $\psi(\theta) \equiv \ln\Psi(\theta)$, $a(\theta; x) \equiv \ln A(\theta; x) = x + \psi(\theta)$ and $b(\theta; x) \equiv \ln(B(\theta)) = \ln\theta + a(\theta; x)$. Thus $a(\theta; x)$ is strictly decreasing and twice continuously differentiable in θ. We will use the term *log-frontier* to refer to the frontier in (a, b) space.

Note that the function $a(\theta; x)$ determines the shape of the frontier, as does the strictly monotonic function $f(a; x)$ for which $b = f(a; x)$. The latter most easily allows us to visualize the shape of the log-frontier, but the former – by expressing the shape of the frontier as a function of θ rather than a – turns out to lead to more elegant results. It is therefore useful to relate the two formulations. Let $s(\theta; x) \equiv -f'(a; x) > 0$ be the *negative* of the slope of the log-frontier as a function of θ. Since $a'(\theta; x) \equiv a'(\theta) = \psi'(\theta)$, $s(.)$ is in fact independent of x, so

$$-f'(a; x) \equiv s(\theta; x) \equiv s(\theta) = -\frac{b'(\theta)}{a'(\theta)} = -\frac{1}{\theta a'(\theta)} - 1. \tag{4}$$

The fact that $s(\theta)$ is independent of x will formally justify the earlier assertion that a change in x represents Hicks neutral technical progress. A change in x represents a shift of the log-frontier along 45° lines, i.e. lines of constant θ. If $\eta(\theta)$ is the elasticity of $s(\theta)$ with respect to θ, the log-frontier will then be concave (convex) where $\eta(\theta) \le 0$ ($\eta(\theta) \ge 0$), with

$$\eta(\theta) \equiv \frac{\theta s'(\theta)}{s(\theta)} = -\frac{1 + \theta\frac{a''(\theta)}{a'(\theta)}}{1 + \theta a'(\theta)} = -\frac{f''(a)}{f'(a)\big(1 - f'(a)\big)}. \tag{5}$$

Note that the elasticity of the slope of the frontier in (A, B) space with respect to θ is simply $\eta(\theta) + 1$.

The first property of the frontier we examine concerns the firm's possibilities to achieve infinite efficiency in any one input through technology choice.

Property 1 *We say that the technology frontier is* ***strongly bounded*** *iff* $A, B < \infty\ \forall\ \theta \ge 0$ *and both* $\lim_{\theta\to\infty} A < \infty$ *and* $\lim_{\theta\to\infty} B < \infty$. *We say that the technology frontier is* ***weakly bounded*** *iff no technology within it exists that gives one input infinite efficiency and the other strictly positive efficiency. In other words the sets* $\{A \mid A > 0$ *and* $\Phi_A(A\mathrm{e}^{-x}) + \Phi_B(B\mathrm{e}^{-x}) \le 0\ \forall\ B \ge 0\}$ *and* $\{B \mid B > 0$ *and* $\Phi_A(A\mathrm{e}^{-x}) + \Phi_B(B\mathrm{e}^{-x}) \le 0\ \forall\ A \ge 0\}$ *are both empty.*

Lemma 1 *The technology frontier is weakly bounded if and only if*

$$\lim_{\theta\to\infty} a(\theta) = \lim_{\theta\to 0} b(\theta) = -\infty. \quad (6)$$

Proof. See appendix. ■

Lemma 1 characterises property 1 in (a, b) space. A weakly bounded frontier is something we may want to impose from economic intuition. However, it is only indirectly related to the question of whether the firm can achieve infinite output through technological choice with finite inputs. When $\rho < 0$, since both inputs are essential to production (see Barro and Sala-i-Martin, 2003, chapter 1.4) a weakly bounded frontier is sufficient but not necessary to preclude a firm from achieving infinite output through technology choice: infinite efficiency is required in both inputs (if finite in quantity) to achieve infinite output. If $\rho > 0$ then a weakly bounded frontier is not sufficient to preclude it: infinite efficiency in either input will imply infinite output so we would need a strongly bounded frontier (as is the case in CC) to ensure an interior solution.

Clearly, we should think of the shape of the technology frontier as an empirical object rather than a theoretical construct. However, we will show below that frontiers which satisfy a distinct but related property that we call *limited convexity* prove to be far easier to work with. The intuition is that as the frontier becomes more convex, an interior choice on the frontier becomes less attractive to the firm. Since the only constraint implied by this property is the intuitively desirable one (discussed above) that technology choice implies that $\sigma_{LR} > \sigma$, we will effectively restrict our attention throughout the paper to frontiers which have limited convexity everywhere.

Property 2 *We say that the technology frontier has* ***limited convexity at θ*** *if and only if*

$$\eta(\theta) < \frac{-\rho}{1-\rho}. \quad (7)$$

We say that the frontier has ***limited convexity everywhere*** *iff the frontier has limited convexity for all* $\theta \geq 0$.

2.1.2 Adjustment costs

An important feature of the paper is the introduction of adjustment costs into the firm's choice of technology. The firm might naturally face implementation costs when introducing a new technology, and might experience a temporary fall in efficiency due to learning costs, while new procedures and techniques are assimilated by the firm. The effect of adjustment costs is to create a difference in the short- and long-run *ES*, which, as discussed above and below, is in line with the evidence.

We could potentially introduce adjustment costs in both a and b but we will take a particularly simple approach to introducing adjustment costs in section 3, which is simply to impose adjustment costs in θ (this simplification is discussed further in section 2.4.1). Regardless of this, the short run – defined theoretically as where adjustment costs preclude any change in the choice of technology – is straightforward to describe. In the short run, from (1), we just have a CES production function with short-run elasticity of substitution $\sigma = \frac{1}{1-\rho}$. We describe the outcome once technology has adjusted completely as the long run. In the remainder of section 2.1 and sections 2.2 and 2.3, we derive a set of theoretical results describing how the shape of the frontier influences this long-run outcome; effectively the outcome in the absence of adjustment costs. The medium run, where partial adjustment has occurred, is best described in the context of a quantitative model. This is done in section 3.

2.1.3 The firm's problem in the absence of adjustment costs

Suppose the factor prices of labour and capital are respectively w and $r+\delta$, and let us denote the capital-labour and factor price ratios by k and Λ,

$$k \equiv \frac{K}{L};\ \Lambda \equiv \frac{w}{r+\delta}. \tag{8}$$

In the absence of adjustment costs, the firm's problem is to choose its position on the technology frontier, θ, and its inputs, K and L, to maximise its profits $Y-(r+\delta)K-wL$, where, implied by (1) and the definition of θ and $a(\theta)$, we now write

$$Y = \left[\mathrm{e}^{\rho a(\theta;x)}K^{\rho} + \theta^{\rho}\mathrm{e}^{\rho a(\theta;x)}L^{\rho}\right]^{1/\rho}. \tag{9}$$

The first order conditions for K and L are, respectively,

$$\mathrm{e}^{a(\theta;x)}k^{\rho-1}\left(k^{\rho}+\theta^{\rho}\right)^{\frac{1-\rho}{\rho}} = r+\delta \tag{10}$$

and

$$\mathrm{e}^{a(\theta;x)}\theta^{\rho}\left(k^{\rho}+\theta^{\rho}\right)^{\frac{1-\rho}{\rho}} = w, \tag{11}$$

which in turn imply,

$$\theta^{\rho}k^{1-\rho} = \Lambda. \tag{12}$$

With a little algebra, the first order condition for θ through technology choice gives

$$\frac{k^{\rho}}{\theta^{\rho}} = s(\theta). \tag{13}$$

Note that, since $s(\theta)$ is independent of x as described in section 2.1.1, x does not enter (12) and (13) and so a change in x will cause no change in the capital-labour ratio k for given factor prices, so the interpretation of x as Hicks neutral progress is valid. As we would expect, the first order conditions (10) and (11) imply the firm makes zero profit in equilibrium due to constant returns to scale in K and L and perfect competition. Of course, this is conditional on equations (12) and (13) characterising the solution to the firm's problem. Proposition 2 establishes sufficient conditions for this. First let Λ^S be the set of values of the factor price ratio Λ for which equations (12) and (13) have a solution for k and θ. Since $s(\theta)$ is continuous, Λ^S must be a convex set. We can then define an interior equilibrium as follows.

Definition: *An **interior equilibrium** is a quartet* $(k^*, \theta^*, r+\delta, w)$ *such that i) the factor price ratio* $\Lambda = {}^{w}/_{r+\delta} \in \Lambda^S$ *and the first order conditions* (10), (11) *and* (13) *are satisfied, and ii) if all firms choose* $\theta = \theta^*$ *and inputs* K *and* L *so that* ${}^{K}/_{L} = k^*$, *no firm can increase its profits by deviating from this choice given factor prices* $r+\delta$ *and* w.

Proposition 2 *has the following parts:*

i) (Uniqueness) A sufficient condition for equations (12) *and* (13) *to have a unique solution* (k^*, θ^*) *for all* $\Lambda \in \Lambda^S$ *is that the frontier has limited convexity everywhere. A necessary condition is the frontier either has limited convexity everywhere or nowhere.*

ii) (Second order conditions) Any solution (k^*, θ^*) *to the first order conditions* (10), (11) *and* (13), *and therefore to* (12) *and* (13), *will locally maximise the firm's profits if the frontier has limited convexity at* θ^*.

iii) (Corner solutions) If $\rho < 0$ *and the frontier has limited convexity everywhere, a sufficient condition for there to be no corner solutions to the firm's problem is that the frontier is weakly bounded. Therefore the unique solution to* (12) *and* (13) *represents the unique equilibrium and an interior one.*

Proof. See appendix. ■

Because of parts *i*) and *ii*) of proposition 2, we henceforth assume that frontier has limited convexity everywhere. Formally, we do not characterise the outcome when the frontier does not have this property. Part *i*) of proposition 2 gives conditions for the left-hand side of equation (12) to be strictly monotonic in k given equation (13). If the frontier has limited convexity on some regions of the frontier and not others, this expression cannot be strictly monotonic in k, and hence there must exist some factor price ratios for which (12) and (13) have multiple solutions.[8]

In order to best understand parts *ii*) and *iii*) of proposition 2, it is perhaps useful to review the standard case without technological choice. We would not have equation (13), and θ would be exogenous in equation (12), giving a unique solution for k. The fact that Y has constant returns to scale in K and L, together with the standard

[8] This raises the possibility of multiple equilibria, where, for example, we might have three solution pairs to the first-order conditions with two satisfying limited convexity and the third not. We do not pursue this line of investigation in this paper.

conditions $Y_{KK}, Y_{LL} < 0$, imply that the Hessian of Y with respect to K and L is everywhere negative semidefinite with one strictly negative and one zero eigenvalue (the zero eigenvalue corresponding to the fact that a constant returns to scale production function only pins down the ratio of the inputs and not their scale). This in turn implies quasi-concavity. Unlike this standard case, proposition 2 does not show that the relevant Hessian is everywhere negative semidefinite; it only shows it is negative semidefinite at any solution (k^*, θ^*) to equations (12) and (13). If this solution is unique, as implied by limited convexity, then only one interior equilibrium is possible. However, if corner equilibria are possible (those in which at least some non-zero measure of firms choose either $\theta = 0$ or $\theta = \infty$, or $k = 0$ or $k = \infty$), then an interior equilibrium might not exist or there might be multiple equilibria. When $\rho < 0$, a weakly bounded frontier implies that any firm choosing either $\theta = 0$, $\theta = \infty$, $k = 0$ or $k = \infty$ will have zero output and therefore negative profits. Hence the unique solution to (12) and (13) represents the unique equilibrium, which is an interior one.

Let us now assume that we have sufficient conditions for the solution to equations (12) and (13) to represent the solution to the firm's problem remembering that this is the (*long-run*) solution in the absence of adjustment costs in technology choice.[9] The following lemma gives the long-run elasticity of substitution σ_{LR} assuming a full adjustment of θ.

Lemma 3 *Suppose the frontier has limited convexity everywhere. If equations* (12) *and* (13) *have a solution* (k^*, θ^*) *that represents the (long-run) solution to the firm's problem (i.e. there are no corner solutions), then the solution has the property that* k *is strictly increasing in* Λ*, and that* θ *is strictly increasing in* Λ *if and only if* $\rho < 0$*. The long-run capital share,* Γ_{LR}*, and long-run capital-labour elasticity of substitution,* σ_{LR}*, are both functions solely of* θ *and are given respectively by*

$$\Gamma_{LR}(\theta^*) \equiv \frac{k}{k + \Lambda} = \frac{s(\theta^*)}{1 + s(\theta^*)} \tag{14}$$

and

$$\sigma_{LR}(\theta^*) = \frac{\rho + \eta(\theta^*)}{\rho + (1 - \rho)\eta(\theta^*)}. \tag{15}$$

Corollary 4 *It follows that:*

i) Limited convexity everywhere implies that the solution satisfies $\sigma_{LR}(\theta^*) > \sigma \equiv \frac{1}{1-\rho}$ *for all* θ^**.*

[9] Note that while proposition 2 characterizes conditions for an interior long-run solution in the absence of adjustment costs, we would often expect the presence of adjustment costs to lead to an interior solution even if there is no interior solution in their absence. In such a case adjustment costs would play a major role in determining the nature of the growth path. We do not explore this scenario in this paper.

ii) *If* $\rho < 0$*, then* $\sigma_{LR} > 1$ *or* $\sigma_{LR} < 1$ *if and only if the log-frontier is locally strictly convex or strictly concave respectively.*

iii) *If* $\rho > 0$*, limited convexity implies* $\sigma_{LR} > \sigma > 1$*.*

Proof. The first order conditions (12) and (13) imply (14) directly. The remainder of the proof is an extension of the proof of part *i*) of proposition 2. Taking logs and total derivatives of (13) implies (see equation (B4) in the appendix)

$$\rho \,\mathrm{dln}\,\theta + (1-\rho)\,\mathrm{dln}\,k = \left\{ \frac{\rho + (1-\rho)\eta(\theta)}{\rho + \eta(\theta)} \right\} \mathrm{dln}\,k. \tag{16}$$

Limited convexity everywhere implies that $\rho + \eta(\theta) < 0$ and that the expression in the curly brackets on the right-hand side of (16) is strictly positive. Therefore from (12), k must be strictly increasing in Λ; since (13) implies that $\rho\,\mathrm{dln}\,k = (\rho + \eta(\theta))\,\mathrm{dln}\,\theta$ it follows that θ is strictly increasing in Λ if and only if $\rho < 0$. The long-run elasticity of substitution $\sigma_{LR} \equiv {}^{\mathrm{dln}\,k}/_{\mathrm{dln}\,\Lambda}$ is also given by (16) and (12). The corollary follows immediately. ■

Lemma 3 implies that the slope of the log-frontier determines the capital share and its curvature determines σ_{LR}. Part *i*) of the corollary confirms the intuition that technology choice increases the elasticity of substitution on frontiers for which we are able to characterize the long-run outcome (i.e. those with limited convexity everywhere) so $\sigma_{LR} > \sigma$. The assumption of limited convexity places no upper bound on σ_{LR} however. As the curvature of the frontier approaches the maximum imposed by limited convexity, we can see that capital and labour approach perfect substitutes, i.e. $\sigma_{LR} \to \infty$. Note also the implications of part *iii*). In a framework such as CC, where the inputs are skilled and unskilled labour and $\rho > 0$ as in CC, we cannot arrive at a frontier which delivers the Cobb-Douglas result ($\sigma_{LR} = 1$). Of course, since there is no evidence that factor shares between skilled and unskilled workers remain constant over time, such a result would not be desirable anyway.

When $\rho < 0$, convexity or concavity of the frontier determines $\sigma_{LR} \geq 1$ or $\sigma_{LR} \leq 1$ respectively. It follows that the long-run production function is Cobb-Douglas iff the log-frontier is linear. In the following sub-section, we extend this to describe the shape of the frontier that produces a general long-run CES production function.

2.2 Long-run CES production functions

Suppose firms face a standard problem, which we denote problem P_1, to choose K and L to maximise $Y - (r+\delta)K - wL$ given factor prices $r + \delta$ and w and a CES production technology with elasticity of substitution $\sigma_{LR} = \frac{1}{1-\mathcal{R}}$ with $\mathcal{R} < 1$:

$$Y = \begin{cases} X\left(\alpha K^{\mathcal{R}} + (1-\alpha)L^{\mathcal{R}}\right)^{\frac{1}{\mathcal{R}}} & \text{when } \mathcal{R} \neq 0 \\ XK^{\alpha}L^{1-\alpha} & \text{when } \mathcal{R} = 0. \end{cases} \tag{17}$$

We assume that the solution to P_1 represents a symmetric equilibrium where all firms choose the same capital-labour ratio $k \equiv K/L$. If $\mathcal{R} > 0$, neither input is essential to production and a firm choosing $L = 0$ or $K = 0$ respectively would obtain output $Y = X\alpha^{\frac{1}{\mathcal{R}}}K$ or $Y = X(1-\alpha)^{\frac{1}{\mathcal{R}}}L$. So if $\mathcal{R} > 0$, a symmetric equilibrium can only hold if

$$X\alpha^{\frac{1}{\mathcal{R}}} \leq r + \delta \tag{18}$$

and

$$X(1-\alpha)^{\frac{1}{\mathcal{R}}} \leq w. \tag{19}$$

Now suppose firms face the current problem incorporating technology choice, problem P_2, which is to choose K, L and θ to maximise $Y - (r+\delta)K - wL$ where, writing $x = \ln X$, Y is given by (9), repeated below for convenience

$$Y = \left[\mathrm{e}^{\rho a(\theta;x)}K^\rho + \theta^\rho \mathrm{e}^{\rho a(\theta;x)}L^\rho\right]^{1/\rho}, \tag{20}$$

where we assume $\mathcal{R} > \rho$. Due to constant returns to scale in K and L, the solutions to problems P_1 and P_2 only determine the output-labour and capital-labour ratios $y \equiv {}^{Y}\!/_{L}$ and $k \equiv {}^{K}\!/_{L}$ rather than the levels of Y, K and L. We can then prove the following proposition.

Proposition 5 *If P_2 has an interior equilibrium solution (which is therefore described by equations* (12) *and* (13) *above) this solution will result in identical outcomes for $y \equiv {}^{Y}\!/_{L}$ and $k \equiv {}^{K}\!/_{L}$ to problem P_1 if and only if the function $a(\theta;x)$ takes the form*

$$a(\theta;x,\mathcal{R}) = \begin{cases} x + \frac{1}{\mathcal{R}\zeta}\ln\left(\alpha^\zeta + (1-\alpha)^\zeta\theta^{-\mathcal{R}\zeta}\right) & \text{when } \mathcal{R} \neq 0 \\ x + \frac{1}{\rho}\left[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right] - (1-\alpha)\ln\theta & \text{when } \mathcal{R} = 0 \end{cases} \tag{21}$$

where the constant $\zeta \equiv \frac{\rho}{\rho-\mathcal{R}}$. Noting that $\zeta \to 1$ as $\mathcal{R} \to 0$ and that $\frac{\partial\zeta}{\partial\mathcal{R}}|_{\mathcal{R}=0} = \frac{1}{\rho}$, it follows by L'Hôpital's rule that $a(\theta;x,\mathcal{R})$ is continuous in $\mathcal{R}$. The functions for the slope and elasticity of the frontier implied by (21) *are:*

$$s(\theta;\mathcal{R}) = \left(\frac{\alpha}{1-\alpha}\right)^\zeta \theta^{\mathcal{R}\zeta}\ ;\ \eta(\theta;\mathcal{R}) = \mathcal{R}\zeta < \frac{-\rho}{1-\rho}. \tag{22}$$

Corollary 6 *It follows that:*

i) If $\rho\mathcal{R} > 0$, then the frontier (21) *is both weakly and strongly bounded; if $\mathcal{R} = 0$, then it is weakly but not strongly bounded; if $\rho\mathcal{R} < 0$ then it is not weakly (and therefore not strongly) bounded.*

ii) If $\mathcal{R} \leq 0$, since the frontier (21) *is weakly bounded and has limited convexity everywhere, then, by proposition 2, P_2 has a unique equilibrium which is an interior one.*

iii) If $\mathcal{R} > 0$ and conditions (18) *and* (19) *hold, then the unique solution to equations* (12) *and* (13) *describes an interior equilibrium for P_2.*

Proof. See appendix. ■

The parameter α effectively serves the role of a tilt parameter, which changes the slope of the frontier and so the capital share. Proposition 5 identifies the shape of the frontier required to produce a long-run CES production function. For both the standard CES framework, P_1, and the model with technology choice, P_2, note the importance of condition (18) when $\mathcal{R} > 0$.[10] La Grandville (2012) argues that a CES production function with an *ES* greater than 1 is incompatible with competitive equilibrium in the presence of any sustained capital-augmenting progress. We can see that if X grows over time and $r + \delta$ is constant, then condition (18) must at some point in the growth path be violated and a symmetric equilibrium can no longer prevail. La Grandville (2012) shows that if K and L are substitutes and there is permanent capital-augmenting technical progress, the long-run growth path must have the awkward property that $r + \delta$ is unbounded over time. Note, however, that this is a general property of CES when $\mathcal{R} > 0$, rather than being related to the derivation of CES via a model of technology choice.

If we had adjustment costs in θ, the frontier (21) would give us a production function which is CES in both the short- and long-run, with the long-run capital-labour elasticity of substitution $\frac{1}{1-\mathcal{R}}$ greater than the short-run $\frac{1}{1-\rho}$. Even abstracting from adjustment costs, technology choice might give us an interesting way to think about the CES production function. Consider, for instance, the literature on the normalization of the CES production function (see La Grandville, 1989, 2008, Klump and La Grandville, 2000, Klump and Saam, 2008, León-Ledesma et al., 2010, and Cantore and Levine, 2012). Suppose, for example, ρ is very low, so individual technologies are almost Leontief, and we are interested in the impact of an increase in the *ES* for some research question. Here, this would correspond to the impact of an increase in the curvature of the frontier keeping ρ fixed. Clearly, there would be no unique way to increase the curvature of the frontier. We might, for example, do it in such a way as to keep one point on the frontier fixed, so the new frontier is tangent to the old one at that point. This would correspond to choosing the point of normalization in the above literature.

2.3 Long-run dynamics

Suppose we have a general frontier. As shown in subsection 2.1.3, the capital share and the long-run elasticity of substitution are given by the slope and curvature of the log-frontier at the point where the firm is located. As the frontier expands outwards, the firm will choose a different position on it. Similarly, a change in capital taxes or

[10] See Akerlof and Nordhaus, 1967, and La Grandville, 2012; see also the conditions for a symmetric equilibrium in the main proposition of CC.

investment-specific technical change will cause a change in the firm's location. In order to describe the long-run evolution of the capital share and elasticity of substitution, we therefore need to track the position of the firm in light of these changes. This is the principal purpose of this section.

2.3.1 The long-run growth path

We first characterize the long-run direction of travel of the firm's technology choice in (a, b) space as the technology frontier expands due to an increase in x, i.e. due to Hicks-neutral technical progess. On the long-run growth path, we neglect any adjustment costs in technology choice and $r_t + \delta$ is held constant.

To make the analysis richer, we introduce capital taxes τ_k and also allow a role for investment specific technical change. Suppose that the capital accumulation equation takes the form $K_{t+1} = Q_t(Y_t - C_t) + (1 - \delta)K_t$ where C_t is consumption and Q_t, the relative price of consumption to investment goods, has a constant growth rate $d \ln Q$. The rental rate of capital, r_t^K, will therefore grow at a long-run rate $-d \ln Q$, holding $r_t + \delta$ constant. We can then combine the first order conditions for K, equation (10), and θ, equation (13), to obtain

$$(1 - \tau_k)\mathrm{e}^{a(\theta;x)} \left(1 + \frac{1}{s(\theta)}\right)^{\frac{1-\rho}{\rho}} = r_t^K. \tag{23}$$

At the beginning of section 2.1.1, we defined the function $\Psi(\theta)$ implicitly by the equation $\Phi_A\big(\Psi(\theta)\big) + \Phi_B\big(\theta\Psi(\theta)\big) = 0$ given the frontier (3), and wrote $a(\theta; x) \equiv x + \psi(\theta)$ where $\psi(\theta) \equiv \ln \Psi(\theta)$. Noting that $a'(\theta; x) = \psi'(\theta)$ and using the formula for $s(\theta)$ from (4), we can then write down the following total derivative,

$$\mathrm{d}\theta = \frac{\mathrm{d}a - \mathrm{d}x}{\psi'(\theta)} = \frac{\mathrm{d}a - \mathrm{d}x}{a'(\theta)} = \big(1 + s(\theta)\big)\,\theta(\mathrm{d}x - \mathrm{d}a). \tag{24}$$

Taking logs of (23) and totally differentiating, we get

$$\mathrm{dln}(1 - \tau_k) + \mathrm{d}a - \frac{1-\rho}{\rho}\frac{s'(\theta)}{\big(1 + s(\theta)\big)\,s(\theta)}\,\mathrm{d}\theta = -\,\mathrm{dln}\,Q \tag{25}$$

and substituting for $\mathrm{d}\theta$ from (24) gives

$$\mathrm{d}a = \frac{-(1-\rho)\eta(\theta)\,\mathrm{d}x + \rho\,\mathrm{dln}\,Q + \rho\,\mathrm{dln}(1 - \tau_k)}{-(1-\rho)\eta(\theta) - \rho} \tag{26}$$

and

$$\mathrm{d}\theta = \frac{-\rho\theta(1 + s(\theta))}{-(1-\rho)\eta(\theta) - \rho}\big(\mathrm{d}x + \mathrm{dln}\,Q + \mathrm{dln}(1 - \tau_k)\big)\,. \tag{27}$$

Since $b(\theta; x) = \ln\theta + a(\theta; x)$, then

$$\begin{aligned} \mathrm{d}b &= \mathrm{d}x + s(\theta)(\mathrm{d}x - \mathrm{d}a) \\ &= \mathrm{d}x + \frac{-\rho s(\theta)\,\mathrm{d}x - \rho s(\theta)\,\mathrm{dln}\,Q - \rho s(\theta)\,\mathrm{dln}(1-\tau_k)}{-(1-\rho)\eta(\theta) - \rho}. \end{aligned} \tag{28}$$

Since the frontier has limited convexity everywhere, the denominators of the ratios on the right-hand sides of (26), (27) and (28) are all strictly positive. For brevity, we restrict ourselves to describing the case where $\rho < 0$, so capital and labour are complements in the individual technologies among which the firm chooses.

Let us consider first the effects of Hicks-neutral progress (an increase in x) assuming $\mathrm{dln}\,Q = \mathrm{dln}(1-\tau_k) = 0$. Since the factor price of capital is held constant along the long-run growth path, Hicks-neutral technical progress has the standard effect of increasing the price of labour relative to the price of capital. We would therefore expect the firm's incentives to choose labour-saving technologies (those with a higher b) to increase. This is verified by equation (28); when $\rho < 0$ we can see that ${}^{\partial b}\!/_{\partial x} > 1$. When $\eta(\theta) > 0$ ($\eta(\theta) < 0$), the log-frontier is strictly convex (concave) and a sacrifice in capital efficiency a leads to greater (lower) gains in labour efficiency; so an increase in x is accompanied by a leftward (rightward) movement along the frontier. Because the firm is already making an optimal decision 'before the change in x,' it is the second order properties (the curvature) of the frontier rather than the first order ones (the slope) that determine the direction of travel.

Now suppose $\mathrm{d}x = 0$ and investment goods are becoming cheaper ($\mathrm{dln}\,Q > 0$) or capital taxes are falling ($\mathrm{dln}(1-\tau_k) > 0$). Both would reduce the price of capital, so we would expect them to have similar qualitative effects, which they do. Note from (26) and (28) that, as expected, the changes in b and a have the opposite sign and their magnitudes are related by the slope of the frontier: when $\mathrm{d}x = 0$, the frontier is static and so we only expect movements along the frontier. A reduction in capital taxes causes capital to become cheaper and encourages the use of labour-saving technologies: therefore the firm moves leftwards along the frontier.

We now consider two examples. The first is the log-frontier shown in figure 3. It comprises two infinitely long linear segments of different slopes, joined by a 'smooth kink.' So long as the left-hand segment is not vertical and the right-hand segment not horizontal, the frontier will be weakly bounded, and provided the curvature of the kink is not too large, limited convexity will be satisfied everywhere along the log-frontier. Hence sufficient conditions for an interior solution will be met. The long-run growth path is also shown on figure 3. Initially a stays constant but then falls while the firm chooses a point on the kink. It then remains constant once we reach the left-hand linear segment of the frontier. Along this growth path there is a secular rise in the capital share as we move to a more steeply sloped part of the log-frontier. Figure 4 shows the growth path when we have Hicks-neutral expansion of a linear (Cobb-Douglas) frontier combined with investment-specific technical change.

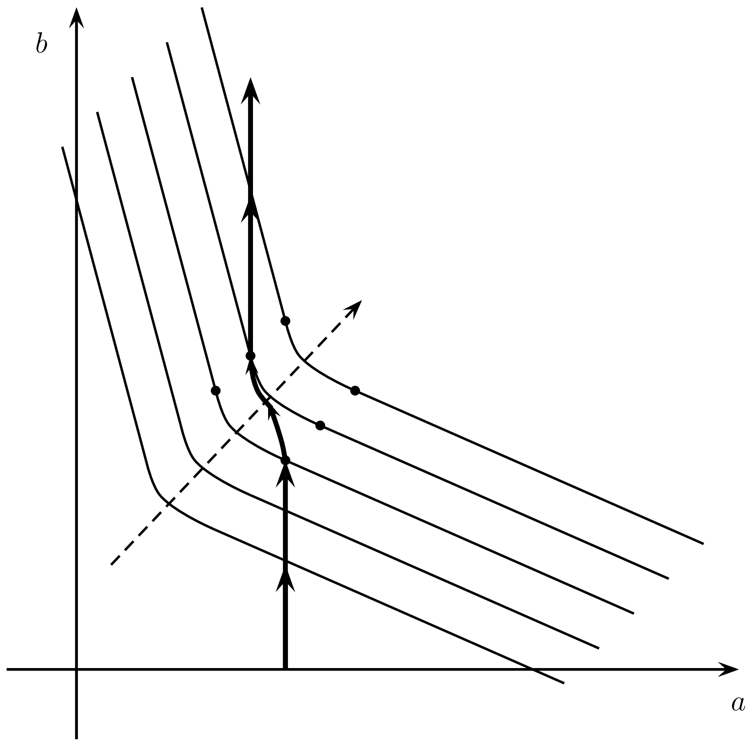


Figure 3: A ‘piece-wise’ linear log-frontier.

The former causes a vertical movement and latter a leftward movement along the frontier, so both combine to give a leftward diagonal growth path which is steeper than the frontier.

2.3.2 Weakly bounded frontiers and the long-run elasticity of substitution

We now present a final result regarding weakly bounded frontiers. We know from the corollary to lemma 3, that when $\rho < 0$ the long-run elasticity of substitution $\sigma_{LR}(\theta) > 1$ if and only if the frontier is strictly convex at θ. However, weakly bounded frontiers must have the property that $\lim_{\theta\to\infty} \sigma_{LR}(\theta) \leq 1$. This is shown in the following proposition.

Proposition 7 *Suppose we have a convex weakly bounded frontier $a(\theta)$such that the limits $\lim_{\theta\to 0} \eta(\theta)$ and $\lim_{\theta\to\infty} \eta(\theta)$ exist. Then it must be the case that*

$$\lim_{\theta\to 0} \eta(\theta) = \lim_{\theta\to\infty} \eta(\theta) = 0.$$

Proof. The proof goes as follows. Suppose $\lim_{\theta\to\infty} \eta(\theta) > 0$. Then there must exist an $\epsilon > 0$ and $T < \infty$ such that $\eta(\theta) > \epsilon$ for all $\theta > T$. Then there must exist a

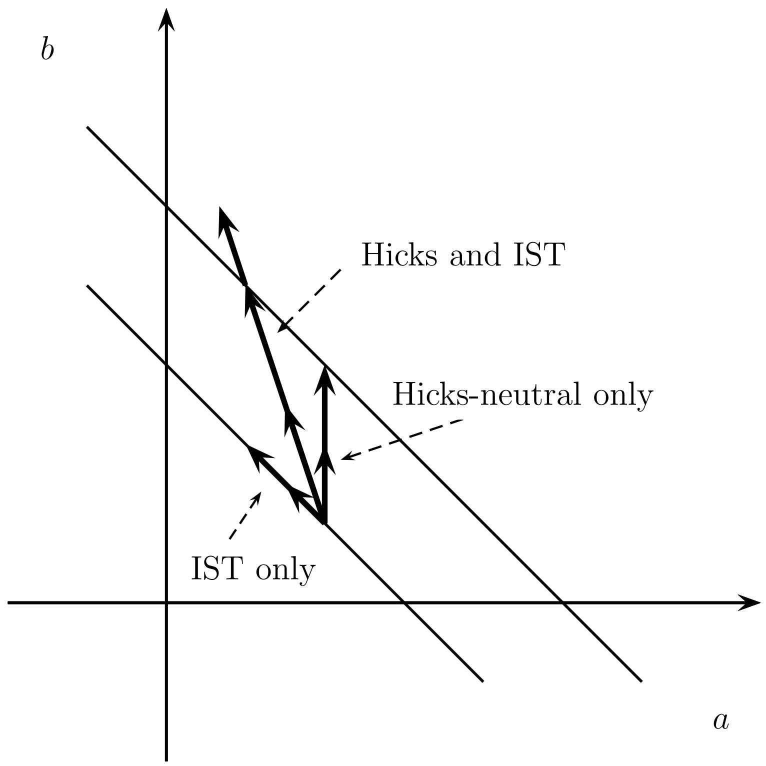


Figure 4: Hicks-neutral expansion of a long-run Cobb-Douglas frontier combined with investment-specific technical progress.

CES frontier $a^{CES}(\theta; x, \alpha, \mathcal{R})$ of the form (21) for some $\mathcal{R} > 0$, x, and α such that $a^{CES}(\theta; x, \alpha, \mathcal{R}) < a(\theta)$ for all $\theta > T$. Since $a^{CES}(.)$ is not weakly bounded by lemma 5 it follows that $a(\theta)$ is not weakly bounded from lemma 1. A similar argument applies if $\lim_{\theta \to 0} \eta(\theta) > 0$. ■

One must be careful how one interprets proposition 7. For any convex weakly bounded frontier, as the frontier expands, θ grows and the long-run capital-labour elasticity of substitution must tend to one. This does not imply, however, balanced growth. We can see this with the simple counter-example below. Suppose the frontier takes the following form,

$$b(a) = \begin{cases} 2x - a & \text{for } a > x \\ c(x-a)^3 + 2x - a & \text{for } a \leq x, \end{cases} \tag{29}$$

where the constant $c > 0$ and where we have expressed $b(.)$ directly as a function of a. The frontier is convex, weakly bounded and, for some t sufficiently low, limited convexity must be satisfied everywhere on the frontier. Hence there will be an interior solution and the elasticity of substitution must tend to one in the long run by proposition 7. Along the strictly convex parts of the frontier, the long-run growth path will follow a leftward trajectory in (a, b) space with b increasing and a falling. Therefore the slope of the frontier at the point of the firm's choice must tend towards $-\infty$ as $x \to \infty$. Therefore the capital share must tend to one by lemma 3. In fact, any weakly bounded frontier which has limited convexity everywhere will have the property that the capital share tends to 1 with continued Hicks-neutral technical progress if its slope $\to -\infty$ as $a \to -\infty$.

2.4 Empirical Applications

Two potential applications of this framework are as follows. Firstly, we might make a set of assumptions that allow us to back out the shape of the frontier. For instance, from a time-series perspective, we might make a 'tape recorder' assumption that the shape of the frontier was fixed through time, so that the long-run growth path takes us to different points on this frontier as it shifts out due to technical progress. With a further set of (stringent) assumptions, we might then be able to back out the shape of the frontier using the results obtained in section 2.3.1.[11]

Secondly, we might assume a shape for the long-run frontier, and explore whether the flexibility that this production function provides in allowing a time-varying elastic-

[11] It would be necessary to make an assumption regarding the nature of technical progress, most likely that technical progress was Hicks-neutral. Assuming one could filter time series data for labour productivity and the capital-labour ratio to obtain an approximate long-run growth path (which here would be defined as the path that labour productivity and the capital-labour ratio would have taken had $r_t + \delta$ been constant) it ought then in principle be possible to use a recursive technique to extract the shape of the frontier using the results of section 2.3.1. Such an exercise is left for future research.

ity of substitution better allows us to model the short- and particularly the medium-run dynamics of the labour share. Given the overall macroeconomic consensus, a reasonable assumption is that the shape of the long-run frontier is compatible with balanced growth – i.e. that it is linear in log-space. It is this empirical exercise that we undertake in section 3, where we model the U.S. labour share.

2.4.1 Short-run dynamics

A key feature of short-run dynamics in this framework is the presence of adjustments costs in changing technologies. As noted in section 2.1.2, we impose adjustment costs only in θ. In terms of simplicity, imposing adjustment costs in only one dimension of the technology space means that technology choice brings only one extra state variable to a model rather than two. It is also a geometrical simplification, since it means the firm is always on the technology frontier if not at its ideal long-run position on it. Nonetheless, there is a trade-off to this simplification. Suppose in (A, B) space, the firm's current choice of technology is (A_1, B_1). An outward shift along the ray from the origin through (A_1, B_1) to (A_2, B_2) (i.e. keeping θ constant) represents TFP or Hicks-neutral growth. Shifting from one ray to another (changing θ) represents a shift in the trade-off of the efficiencies with which capital and labour are used. Both of these processes might naturally involve adjustment costs, due to costs in learning and implementation. In the former, adjustment costs might also arise in the form of slow diffusion processes for technology innovations. Since we do not impose adjustment costs in the former, we may fail to capture such features.[12]

Suppose a change in θ implies a loss of output $\varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)Y$ where $\varphi \geqslant 0$, $\varphi(1) = \varphi'(1) = 0$ and $\varphi''(.) > 0$. Given factor prices for capital and labour $r_t + \delta$ and w_t respectively, the firm chooses θ_t, K_t and L_t to maximise

$$\sum_{t=0}^{\infty}\left\{\left[\prod_{s=0}^{t}\left(\frac{1}{1+r_s}\right)\right]\left[Y_t\left(1-\varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)\right)-(r_t+\delta)K_t-w_tL_t\right]\right\} \tag{30}$$

where

$$Y_t = \left(e^{\rho a(\theta_t;x_t)}K_t^{\rho} + \theta_t^{\rho}e^{\rho a(\theta_t;x_t)}L_t^{\rho}\right)^{\frac{1}{\rho}}. \tag{31}$$

In section 2.3.1, we introduced a role for investment-specific technical change in terms of Q_t, the relative price of consumption goods to investment goods. In modelling short- or medium-run fluctuations, there are other forms of technology shocks we could potentially introduce into (31), depending on the focus of the relevant application. It is possible that the quality of the inputs improves exogenously regardless of technology choice, so we could for example replace L_t by Z_tL_t, where Z_t might capture exogenous

[12] However, we may still be able to partly capture these features, albeit much more crudely, in the dynamic properties of the shocks themselves: for example, by making changes in x_t more gradual.

changes in human capital. If we replaced θ_t by ${}^{\theta_t}/_{\bar{\theta}_t}$ permanent shocks to $\bar{\theta}_t$ would serve as a 'tilt' shock, playing a similar role to the distributional shocks of Ríos-Rull and Santaeulàlia-Llopis (2010). Thus we would replace (31) with

$$Y_t = X_t \left(e^{\rho a\left(\theta_t/\bar{\theta}_t; x_t\right)} K_t^\rho + \left(\frac{\theta_t}{\bar{\theta}_t}\right)^\rho e^{\rho a\left(\theta_t/\bar{\theta}_t; x_t\right)} (Z_t L_t)^\rho \right)^{\frac{1}{\rho}}. \tag{32}$$

Arguably technology choice gives the distributional shock $\bar{\theta}_t$ a more natural interpretation than shocks to the exponent of a Cobb-Douglas production function.

2.4.2 The production function with long-run Cobb-Douglas

We can see from proposition 5, that we arrive at long-run Cobb-Douglas, guaranteeing balanced growth, if the frontier takes the form

$$a(\theta; x_t', r) = x_t' + \frac{1}{\rho}\left[\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right] - (1-\alpha)\ln\theta. \tag{33}$$

Replacing x_t' with $x_t = x_t' - \frac{1}{\rho}\left[\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right]$, writing $X_t = e^{x_t}$ and substituting in (34), we get

$$Y_t = X_t \left(\left(\frac{\theta_t}{\bar{\theta}_t}\right)^{\rho(\alpha-1)} K_t^\rho + \left(\frac{\theta_t}{\bar{\theta}_t}\right)^{\rho\alpha} (Z_t L_t)^\rho \right)^{\frac{1}{\rho}}. \tag{34}$$

The relevant first order conditions are then:

$$\left\{1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)\right\} X_t^\rho \left(\frac{Y_t}{K_t}\right)^{1-\rho} \theta_t^{(\alpha-1)\rho} = r_t + \delta. \tag{35}$$

$$\left\{1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)\right\} X_t^\rho \left(\frac{Y_t}{L_t}\right)^{1-\rho} \theta_t^{\alpha\rho} = w_t. \tag{36}$$

$$\begin{aligned} &\alpha\left\{1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)\right\} - \frac{(r_t+\delta)K_t}{Y_t} \\ &-\frac{\theta_t}{\theta_{t-1}}\varphi'\left(\frac{\theta_t}{\theta_{t-1}}\right) - \frac{1}{1+r_t}\frac{\theta_{t+1}}{\theta_t}\varphi'\left(\frac{\theta_{t+1}}{\theta_t}\right)\frac{Y_{t+1}}{Y_t} = 0. \end{aligned} \tag{37}$$

Only equation (37) is non-standard. We can see from the first term in equation (37) that once adjustment takes place, the capital share will equal α. We now use the above production function to model the U.S. labour share. The distributional shock $\bar{\theta}_t$ is potentially useful in this exercise. However, since we are able to obtain a good match for the dynamics of the labour share, we restrict ourselves to using the more conventional technology shocks which are widely used in DSGE models, and analyze how far they can go in matching data properties.

3 The short and medium run dynamics of the labour share

The production technology of section 2 has a wide range of applications in macroeconomics and growth models where the value of the elasticity of substitution matters for both cyclical and long-run phenomena.[13] As discussed above, some of its implications could be used to estimate technology frontiers. Our focus here is on macroeconomic models of fluctuations. These models typically embody a concept of balanced growth in steady state that is required to apply existing solution methods. We analyze a model of short- and medium-run fluctuations of the labour income share. Because of this, we limit ourselves to the specific case in which $\sigma < \sigma_{LR} = 1$, such that the long run production function is Cobb-Douglas. This is possibly the more relevant case for standard dynamic general equilibrium models.

The observed decline in the relative price of investment goods points towards the importance of investment-specific permanent technology shocks as a source of economic fluctuations (Greenwood et al., 1997, 2000 and Fisher, 2006). Permanent investment-specific shocks, however, require the restrictive assumption of a Cobb-Douglas functional form for the production function, preventing any fluctuations in factor shares. The combination of investment-specific permanent shocks with the observed volatility of factor income shares (see Ríos-Rull and Santaeulàlia-Llopis, 2010) presents a puzzle in macroeconomics: if we allow for biased technological progress and require long-run balanced growth we have to restrict the production function to the Cobb-Douglas case, which is not compatible with the observed volatility of factor shares.[14]

Thus, our focus here is on the introduction of technology choices in an otherwise standard Real Business Cycle (RBC) model where we add a rich structure of permanent (potentially factor-biased) technology shocks. We show that the behavior of the labour share generated by the model matches its observed counter-cyclicality in the short run and almost a-cyclicality in the medium run. As Beaudry (2005) points out, it is likely that the choice of production technologies will have important consequences for medium run phenomena. We hence abstract from some of the rigidities and frictions emphasized by the business cycle literature. This also helps keep the model simple in order to highlight the role of factor substitution and biased technical change, and the implementability of the approach. The model presented below also nests a standard RBC model with Cobb-Douglas production function, and an RBC

[13] For instance, the analysis of factor-bias in cross-country technology differences (Caselli and Coleman, 2006), the production of "appropriate technologies" and their impact on cross-country income gaps (Acemoglu and Zilibotti, 2001 and Basu and Weil, 1998), the response of hours to technology shocks (Cantore et al, 2014), and the effect of taxes on factor shares (Chirinko, 2002).

[14] As mentioned earlier, factor income shares fluctuations appear to be driven also by factors unrelated to cyclical fluctuations in mark-ups. As we will see below, even at medium-run frequencies when wage rigidities are absent, we observe substantial fluctuations in the labour share.

model with CES, which allows us to compare the performance of the models in terms of data moments.

Table 1 presents the behavior of some key macroeconomic aggregates using quarterly data for the US for the 1948Q1:2013:Q3 period. The way we filter the data follows Comin and Gertler (2006). They distinguish between short and medium run frequencies of the data to capture medium-run cycles. The *medium run cycle* is obtained using a band pass filter that includes frequencies between 2 and 200 quarters, i.e. it filters the data using a very smooth nonlinear trend. The medium run cycle is made up of a *short run component* (frequencies between 2 and 32 quarters) and a *medium run component* (frequencies between 32 and 200 quarters). Since most of the data are nonstationary in levels, we applied the filters to the growth rate of the series and reconstructed the filtered levels using their cumulative sum.[15] Note that, although the filter includes frequencies of 200 quarters, in the time domain, this translated into cycles of around 10-12 years as in Comin and Gertler (2006). In the results below, we present the standard deviation of the variables relative to output, the correlation with output, and the 95% GMM confidence intervals for this correlation.

The construction of the data follows standard procedures in the literature.[16] Output (Y) was measured as output of the non-farm business sector over civilian non-institutionalized population, consumption (C) is real non-durable and services consumption over civilian non-institutionalized population, investment (Inv) is real private fixed investment plus durable consumption over civilian non-institutionalized population, wages (W) are compensation per hour in the non-farm business sector, and hours worked (L) are measured as all hours in the non-farm business sector over civilian non-institutionalized population. Labour productivity ($Prod$) is measured as Y/L. The labour share measure is important for our exercise. However, measuring the labour share of income is complicated by problems related to how certain categories of income should be imputed to labour and capital owners. Appendix E contains a more thorough discussion of the measures of the labour share we used and their construction. Following Gomme and Rupert (2004), we present three different measures. The first ($LSH1$) is the labour share of income in the non-farm business sector as reported by the Bureau of Labor Statistics. The second ($LSH2$) is the labour share of the domestic corporate non-financial business sector, which is calculated as one minus corporate profits and interests net of indirect taxes over value added. The third measure ($LSH3$) also follows Gomme and Rupert (2004) and calculates the labour share as unambiguous labour income over unambiguous capital income plus unambiguous labour income. We also obtained quarterly data for the labour share for Australia, Canada, The Netherlands, Spain, and the UK reported in Table 2. The countries were chosen on the basis of data availability, and supplementary appendix B gives details

[15]For stationary variables this procedure and filtering directly the level series yielded virtually the same results.

[16]See appendix E for details and sources.

of the sources and data construction.

Table 1: US data moments, 1948Q1:2013:Q3

	Short-run cycle				Medium run cycles (CG filter)			
	Std/Std(Y)	Cor(X,Y)	95% CI		Std/Std(Y)	Corr(X,Y)	95% CI	
C	0.370	0.799	0.746	0.866	0.803	0.837	0.837	0.877
Inv	2.101	0.884	0.835	0.933	2.070	0.918	0.918	0.937
W	0.441	0.166	0.056	0.364	0.750	0.656	0.656	0.708
L	0.853	0.866	0.827	0.904	0.902	0.574	0.574	0.734
Prod	0.490	0.452	0.401	0.654	0.722	0.664	0.664	0.766
LSH1	0.469	-0.243	-0.398	-0.093	0.335	0.203	-0.025	0.432
LSH2	0.590	-0.416	-0.521	-0.247	0.434	0.181	-0.049	0.412
LSH3	0.481	-0.426	-0.533	-0.319	0.410	0.362	0.181	0.543

Table 2: Other countries: labour share moments

		Short run cycle				Medium run cycles (CG filter)			
	Sample (Q)	Std/Std(Y)	Cor(X,Y)	95% CI		Std/Std(Y)	Cor(X,Y)	95% CI	
Australia	1960:Q1-2010:Q4	0.729	-0.382	-0.542	-0.240	0.331	-0.245	-0.432	-0.058
Canada	1981:Q1-2010:Q4	0.894	-0.555	-0.735	-0.374	0.625	-0.430	-0.644	-0.215
Netherlands	1988:Q1-2013:Q3	1.148	-0.668	-0.815	-0.521	0.272	-0.463	-0.713	-0.213
Spain	1995:Q1-2010:Q4	0.672	-0.232	-0.612	0.148	0.308	-0.313	-0.690	0.064
UK	1960:Q1-2010:Q4	0.999	-0.507	-0.663	-0.351	0.539	0.277	0.099	0.454

The behavior of consumption, investment and labour market variables is standard, and the short- medium-run split displays the same behavior as that reported in Comin and Gertler (2006). The standard deviations of all variables, except for investment, increase in the medium run relative to output. Correlations with output also increase with the exception of hours worked, which become less pro-cyclical in the medium run. The labour share displays a clear counter-cyclical behavior at short run frequencies, with a standard deviation that is about 50% that of output for the US, and even larger for the other countries in the sample. In the long run, if balanced growth holds, factor income shares should display no variation or correlation with output. We observe that, at medium-run frequencies, the standard deviation of the labour share falls, and its correlation with output becomes positive, although not significant for two out of the three measures. For the rest of the countries, the standard deviations of the labour share also fall when compared to short-run frequencies, and the medium-run counter-cyclical behavior becomes milder. In the case of the UK, the medium-run correlation with output is positive, but only marginally significant. Overall, hence, we observe that the business cycle counter-cyclical behavior of the labour share tends to fade at medium run frequencies, as does its volatility, indicating a process of convergence towards balanced growth in the long run. We use these results as a benchmark for the models developed next.

3.1 The model

We use a standard RBC model with optimizing representative households and firms. Households maximize their lifetime utility defined over their stream of consumption and leisure, and firms maximize profits. We use a decentralized version of the model where households own the capital and rent it out to firms. The only difference with the standard setting is that firms choose not only capital and labour, but also their technology blueprint (subject to adjustment costs) and the production function is CES in the short run. We present a general version of the model in which we have three types of *permanent* technology shocks: labour-augmenting shocks (Z_t), Hicks-neutral shocks (X_t), and investment-specific shocks (Q_t). Of these three shocks, X_t and Q_t are not compatible with balanced growth if the elasticity of substitution differs from unity. The model then nests other versions which we will use for comparison with data moments, including a standard RBC model with Cobb-Douglas technology, and an RBC model with short and long run CES. For the sake of space, we skip some of the detail for the explanation of the standard parts of the model.

Households choose consumption (C_t), hours worked (L_t), capital stock (K_{t+1}) and one-period non-contingent bonds (B_{t+1}) to maximize their expected lifetime utility $U(\cdot)$:

$$\underset{C_t,L_t,K_{t+1},B_{t+1}}{\text{Max}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t\right),$$

subject to the budget constraint

$$C_t + I_t + B_{t+1} = r_t^K K_t + w_t L_t + (1 + r_t) B_t, \tag{38}$$

and the law of motion for capital:

$$K_{t+1} = (1 - \delta) K_t + Q_t I_t. \tag{39}$$

I_t is investment in new capital stock, r_t^K is the rental rate of capital, r_t is the interest rate on one-period bonds, w_t are wages, and δ is the rate of depreciation of capital. Investment-specific technical change enters the capital accumulation equation by increasing the productivity of new investment goods. In this model, Q_t is also the inverse of the price of investment relative to consumption goods. We specify a utility function separable in consumption and labour:

$$U_t = \log C_t - \upsilon \frac{L_t^{1+\mu}}{1+\mu}, \tag{40}$$

where μ is the inverse of the Frisch elasticity, and υ is a shift parameter.

The firm's problem is to choose K_t, L_t, and the technology "blueprint" θ_t to maximize the present value of the expected future stream of profits subject to the technology constraint given by the production function and the adjustment costs to a change in technology, $\varphi\left(\frac{\theta_t}{\theta_{t-1}}\right)$ as in equation (30), which we reproduce here for convenience:

$$E_0 \sum_{t=0}^{\infty} \left\{ \left[\prod_{s=0}^{t-1} \left(\frac{1}{1+r_s} \right) \right] \left[Y_t \left(1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}} \right) \right) - r_t^K K_t - w_t L_t \right] \right\}, \tag{41}$$

subject to production function (34) where, as mentioned, we abstract from distributional shocks:

$$Y_t = X_t \left((\theta_t^{\alpha-1} K_t)^{\rho} + (\theta_t^{\alpha} Z_t L_t)^{\rho} \right)^{\frac{1}{\rho}}. \tag{42}$$

The law of motion for technology shocks is given by:

$$d\log Z_t = (1-\kappa_Z)\nu_Z + \kappa_Z d\log Z_{t-1} + (1-\kappa_Z)\epsilon_Z, \tag{43}$$

$$d\log X_t = (1-\kappa_X)\nu_X + \kappa_X d\log X_{t-1} + (1-\kappa_X)\epsilon_X, \tag{44}$$

$$d\log Q_t = (1-\kappa_Q)\nu_Q + \kappa_Q d\log Q_{t-1} + (1-\kappa_Q)\epsilon_Q, \tag{45}$$

so that technological progress is specified as (permanent) rate of growth shocks with drifts ν_i and persistence parameters κ_i for $i = Z, X, Q$. The innovations ϵ_i are zero mean normally distributed with covariance matrix Σ. This specification nests the pure random walk when $\nu_i = 0$, $\kappa_i = 0$, and all the off-diagonal elements of Σ are zero.

Defining $\Omega_t = \frac{\theta_t}{\theta_{t-1}}$ and dropping the expectations operator from forward-looking variables for notational convenience, the first order conditions for households and firms yield:

$$\frac{C_{t+1}}{C_t} = \beta(1 + r_{t+1}), \tag{46}$$

$$w_t = \upsilon h_t^{\mu} C_t, \tag{47}$$

$$K_{t+1} = Q_t \left((1 - \varphi(\Omega_t)) Y_t - C_t \right) + (1-\delta) K_t, \tag{48}$$

$$1 + r_{t+1} = \frac{(1-\delta) Q_t}{Q_{t+1}} + r_{t+1}^K Q_t, \tag{49}$$

$$Y_t = X_t \left((\theta_t^{\alpha-1} K_t)^{\rho} + (\theta_t^{\alpha} Z_t L_t)^{\rho} \right)^{\frac{1}{\rho}}, \tag{50}$$

$$\left\{ 1 - \varphi(\Omega_t) \right\} \left(\theta_t^{\alpha-1} X_t \right)^{\rho} \left(\frac{Y_t}{K_t} \right)^{1-\rho} = r_t^K, \tag{51}$$

$$\left\{1-\varphi\left(\Omega_t\right)\right\}\left(\theta_t^{\alpha} X_t Z_t\right)^{\rho}\left(\frac{Y_t}{L_t}\right)^{1-\rho}=w_t, \tag{52}$$

$$\alpha\left\{1-\varphi\left(\Omega_t\right)\right\}-\frac{r_t^K K_t}{Y_t}-\left\{\Omega_t \varphi'\left(\Omega_t\right)-\frac{\Omega_{t+1}}{1+r_t} \varphi'\left(\Omega_{t+1}\right) \frac{Y_{t+1}}{Y_t}\right\}=0. \tag{53}$$

An equilibrium in this context is a set of decision rules $x_t = x(K_t, Z_t, X_t, Q_t)$ for $x_t = \{C_t, L_t, K_{t+1}, B_{t+1}, \theta_t\}$ such that (46)-(53) are satisfied. The model is then appropriately stationarised by dividing all trended variables $K_t, C_t, Y_t, w_t, \theta_t$ by their stochastic trends. The trend for K_t, C_t, Y_t, w_t is defined by $\bar{Y}_t = Z_t X_t^{\frac{1}{1-\alpha}} (Q_{t-1})^{\frac{\alpha}{1-\alpha}}$, whereas the trend for θ_t is given by $\bar{\Theta}_t = (X_t Q_{t-1})^{\frac{1}{1-\alpha}}$.

The functional form for the technology adjustment costs is assumed to be a symmetric exponential function[17] $\varphi(\theta_t/\theta_{t-1}) = 1-\mathrm{e}^{-\frac{1}{2}\tau(\theta_t/\theta_{t-1}-1)^2}$. Parameter τ determines the speed of adjustment. The model then nests a standard RBC with Cobb-Douglas when $\tau = 0$, and an RBC model with CES production function as $\tau \to \infty$. Note that, in the latter case, only the Z_t process can be allowed to contain stochastic or deterministic trends. Given the observed decline in the relative price of investment, forcing Q_t to be temporary is in fact counterfactual.

3.2 Calibration and estimation

In order to simulate the model, we obtain parameter values by a combination of calibration and estimation. We calibrate those parameters for which we can obtain an observable steady state condition or use information from previous studies, and estimate the rest of the parameters. Table 3 presents the calibration values in the first eight rows. We used a standard value for the steady state capital share of 0.33. β is set to 0.99 as we are matching quarterly data, whereas the depreciation rate is a standard 2.5% per quarter. Parameter μ is set to 0.33, which is consistent with macroeconomic estimates of a Frisch elasticity between 2 and 4 (see Peterman, 2012 and Chetty, 2009). For the parameters driving the law of motion of investment-specific technical change, we estimated equation (45) using data for the relative price of investment goods for the 1948Q1-2013:Q3 period. The data were obtained using the implicit deflator for fixed investment and durable goods over the price deflator for non-durables and services consumption. All data were obtained from the BEA. We estimated a drift coefficient of 0.0018 per quarter, a persistence of 0.266, and a standard deviation of the residual of 0.0067.

The values used for the *short-run* elasticity of substitution ranged from 0.1 to 0.3 (implying a ρ coefficient between -9 and -2.3). Time series estimates of the elasticity

[17]Other functional forms such as quadratic adjustment costs did not change the results quantitatively.

Table 3: Calibrated parameters and priors

α	0.33
μ	0.33
β	0.99
δ	0.025
ρ	-9/-2.3
κ_Q	0.266
ν_Q	0.0018
$stdv(\epsilon_Q)$	0.0067
Priors	
$stdv(\epsilon_i)$	InvGamma(0.001, 1)
ν_i	Normal(0.005, 0.01)
κ_i	Beta(0.1, 0.1)
τ	Gamma(10, 5)

of substitution for the US range from 0.4 to 0.7 (see León-Ledesma et al. 2010 and 2014). In our context, these estimated elasticities would be capturing the average value during the adjustment towards unity in the long run. Hence, the short run elasticity must be below these benchmark values. In order to test this effect, we simulated the model with a short run elasticity of 0.2 and an adjustment speed coefficient $\tau = 20$.[18] We then simulated time-series for the data and estimated the elasticity of substitution using an OLS regression on the log first order condition for labour. The estimated value for the elasticity of substitution was 0.6, which is comfortably in the range of estimates from previous studies. Hence, low values in the range of 0.1-0.3 for the short-run elasticity of substitution are consistent with the estimates offered in the literature.

The rest of the parameters were estimated using Bayesian likelihood methods based on the state-space representation of the model and now standard in the DSGE literature. The bottom part of Table 3 presents the prior distributions used to obtain posterior modes using MCMC methods. The priors are standard in the estimated DSGE literature. The standard deviation of shocks (other than ϵ_Q) follows an Inverse Gamma distribution as they are bounded below by zero and unbounded above, drifts follow a Normal distribution, and persistence coefficients follow Beta distributions as they are restricted to the open unit interval. The prior for the adjustment speed τ is drawn from a Gamma distribution as it excludes negative draws.

We estimated different versions of the model, allowing for a variety of shocks (see section 3.3 below). All the models but one contained combinations of two technology shocks and hence we only used a maximum of two observables given that non-

[18] A τ of 20 is a common value obtained from the estimates discussed below.

singularity requires the same number of observables and shocks.[19] The observables we used were the first difference of the log of labour productivity ($d\log(Prod_t)$), and hours per-capita (L_t) when the model contained two shocks. We also used alternative variables as observables such as the growth rate of output per capita, consumption growth, and investment. However, the results remained robust to the choice of observable variables. Given that we estimated a large number of models, we do not report here all the estimation results.[20] It suffices to mention that, in the overwhelming majority of cases, posteriors appeared to be far from the priors, indicating that data are adding relevant information in the estimation process. Standard deviations of shocks vary by specification, but generally match well the volatility of output. Drift parameters are also consistent with the average rate of growth of per capita variables in the economy. Finally, the estimated posterior mode for τ was generally found to be close to 20.[21]

For the RBC model with Cobb-Douglas, the estimation follows the same procedure, but τ is restricted to be zero. For the model with CES only, however, since we cannot have permanent Hicks-neutral and investment-specific technical change, these were specified as AR(1) processes and their persistence and standard deviations estimated as above. Finally, for the CES only models, we calibrated the elasticity of substitution to 0.5, as there is no adjustment towards Cobb-Douglas.

3.3 Model behavior and comparison

Synthetic data for the macroeconomic variables considered were generated using calibrated values and posterior modes of the estimated parameters. We simulated 2,000 data points and kept the last 260, a similar sample size to that in the data. We then applied the same data filters, so we can consistently compare the short and medium run moments with those in the data. Given the large number of models to compare, we discuss here the following set of representative models:

1. RBC1: a standard RBC model with Cobb-Douglas and only labour-augmenting shocks.
2. RBC2: an RBC model with Cobb-Douglas and both labour-augmenting and investment-specific shocks.
3. CES1: a model with CES technology but no technology choice (short and long run CES) with only labour-augmenting shocks.
4. CES2: a model with short and long run CES with permanent labour-augmenting and temporary investment-specific shocks.

[19] We do not add ad hoc measurement errors.

[20] A complete set of results, data, and codes are available upon request.

[21] A value of 20 implies that a 1% change in θ_t incurs a reasonable output cost of 0.1%.

5. CES3: a model with short and long run CES with a permanent labour-augmenting shock and a temporary Hicks-neutral shock.

6. CESCD1: a model with technology choice and only permanent Hicks-neutral shocks.

7. CESCD2: a model with technology choice and permanent labour-augmenting and investment-specific shocks.

8. CESCD3: a model with technology choice and permanent labour-augmenting and Hicks-neutral shocks.

9. CESCD4: a model with technology shocks and permanent Hicks-neutral and investment-specific shocks.

Tables 4 to 8 present the moments calculated with the simulated data for the different models. In all the results below for the CESCD models, we used a short-run elasticity of substitution of 0.2 ($\rho = -4$). Italics denote statistically insignificant correlations using 95% GMM confidence intervals. Models RBC1 and 2 reflect standard results in the literature. Consumption volatility and its correlation with output are higher than in the data, whereas the medium-run fall in the volatility of investment is higher than observed in the data, especially for RBC1. Labour market moments reflect the standard puzzle that the model generates a high volatility of wages and low volatility of hours. Of course, RBC1 and 2 generate no variability of the labour share.

A natural benchmark is to compare the performance of models featuring only one permanent shock and then models with two shocks. Comparing models RBC1, CES1, and CESCD1, we can see that CESCD1 does better at matching investment and consumption in the medium run. It also does a better job at matching wages and hours, although the standard labour market puzzle remains. None of the models is good at matching productivity at both frequencies. Regarding the labour share, CES1 matches better the short-run volatility but generates too strong a negative correlation with output. CESCD1 also generates a strong negative short-run correlation but does slightly better in the medium run where it generates a positive but not significant correlation with output. Overall, CES1 and CESCD1 outperform the basic RBC with Cobb-Douglas and only permanent labour-augmenting shocks. CESCD1 does slightly better at matching the labour share.

We turn now to the models with two shocks. CES2 and CES3 generate some counterfactual results such as the negative (but insignificant) correlation of consumption and output at medium-run frequencies for CES2, or excess short-run volatility of investment. In the medium run, both models produce a more than three-fold reduction in investment volatility. The volatility of hours increases relative to the RBC models in the short run, but in the medium run it goes to almost zero in both CES2 and

CES3. As for the labour share, CES2 and CES3 generate a low short-run volatility and too high a negative correlation. In the medium run, both models do better as the negative correlation falls to values similar to those observed in countries other than the US, whereas the standard deviation relative to output goes to almost zero. Model CES2, unlike RBC2 and the CESCD models, however, has the added limitation that it cannot generate permanent changes in the relative price of investment goods.

Overall, thus, for models featuring either one or two shocks, the models with short- and long-run CES do not do a good job at matching the behaviour of the labour share. Those that do better such as CES3 do it at a substantial cost in terms of matching the behaviour of expenditure components and the labour market. Models with short- and long-run Cobb-Douglas, by definition, cannot generate cyclical fluctuations in the labour share.

The models that do a better job at matching the behavior of the labour share are the CESCD models. Specifically, models CESCD2 and 4 do a very good job at matching the short-run volatility and cyclicality of the labour share. At medium-run frequencies, the correlation falls and becomes insignificant in two cases, and the volatility also falls substantially. Importantly, this is done at a lower cost than the CES models in terms of other data moments, such as consumption, investment, and the labour market. The exception is CESCD2, which generates a high volatility of productivity and a negative correlation of hours and output in the short run. The reason is that, as shown by Cantore et al. (2014), an RBC model with a sufficiently low elasticity of substitution can generate negative responses of hours to a labour-augmenting technology shock. These shocks have a substitution and an output effect on labour demand such that, for a sufficiently low elasticity, the negative substitution effect can outweigh the positive output effect. If this shock dominates, the unconditional correlation can also fall and become negative.[22] In general, CESCD models do the best job at fitting labour share moments, and only slightly worse at fitting some of the labour market moments in the short run. The lack of success at reproducing labour market moments is, however, common to most standard macroeconomic models without frictions. The introduction of other mechanisms such as indivisible labour or search and matching frictions should improve all models' moments, but is beyond the scope of our illustration.[23]

[22]Adding demand shocks such as preference shocks to the coefficient of hours in the utility function would eliminate this negative correlation. However, we have deliberately kept the models simple to highlight the role of the supply side.

[23]We introduced indivisible labour in several versions of the model following Hansen (1985). Indivisible labour increases the volatility of hours and reduces the volatility of wages as expected. But, in the models with choice of technique, the short- and medium-run behavior of the labour share remains consistent with the data. This is a promising avenue for future research as it would help matching not only the labour share, but the *joint* cyclical behavior of its components.

Table 4: Theoretical moments using posterior modes: RBC1 and RBC2

	RBC1 (Z shock)				RBC2 (Z and Q shock)			
	Short-run		Medium run		Short-run		Medium run	
	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)
C	0.604	0.977	0.983	0.997	0.724	0.800	0.880	0.937
Inv	2.269	0.985	1.085	0.977	2.612	0.867	1.782	0.857
W	0.699	0.990	0.987	0.998	0.757	0.902	0.898	0.966
L	0.323	0.954	0.060	0.247	0.455	0.696	0.265	0.497
Prod	0.699	0.990	0.987	0.998	0.758	0.902	0.898	0.967
LSH	0	0	0	0	0	0	0	0

Italics indicate not significantly different from zero using GMM 95% confidence intervals.

Table 5: Theoretical moments using posterior modes: CES1 and CES2

	CES1 (permanent Z shocks)				CES2 (permanent Z temporary Q shocks)			
	Short-run		Medium run		Short-run		Medium run	
	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)
C	0.363	0.820	0.984	0.994	0.824	*-0.044*	0.969	0.992
Inv	2.953	0.982	1.094	0.966	4.407	0.863	1.151	0.96
W	0.392	0.864	0.985	0.995	0.651	0.136	0.971	0.994
L	0.130	0.961	0.020	0.204	0.678	0.556	0.039	0.302
Prod	0.876	0.999	0.996	1.000	0.840	0.741	0.989	0.999
LSH	0.361	-0.955	0.052	-0.195	0.370	-0.927	0.053	-0.284

Italics indicate not significantly different from zero using GMM 95% confidence intervals.

Table 6: Theoretical moments using posterior modes: CES3 (permanent Z temporary X shocks)

	CES3 (permanent Z temporary X shocks)			
	Short-run		Medium run	
	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)
C	0.232	0.645	0.991	0.987
Inv	3.336	0.990	1.148	0.924
W	0.314	0.888	0.990	0.988
L	0.441	0.892	0.040	*0.073*
Prod	0.639	0.950	0.998	0.999
LSH	0.248	-0.848	0.075	*-0.159*

Italics indicate not significantly different from zero using GMM 95% CI's.

Table 7: Theoretical moments using posterior modes: CESCD1 and CESCD2 (technology choice models)

	CESCD1 (permanent X shocks)				CESCD2 (permanent Z and Q shocks)			
	Short-run		Medium run		Short-run		Medium run	
	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)
C	0.640	0.991	0.787	0.974	0.865	0.814	0.870	0.986
Inv	2.119	0.992	1.778	0.953	2.409	0.788	1.486	0.957
W	0.687	0.992	0.836	0.982	0.764	0.812	0.895	0.989
L	0.165	0.963	0.203	0.897	0.722	-0.385	0.217	0.423
Prod	0.842	0.999	0.823	0.994	1.441	0.887	0.929	0.977
LSH	0.131	-0.805	0.070	*0.066*	0.684	-0.648	0.164	*-0.091*

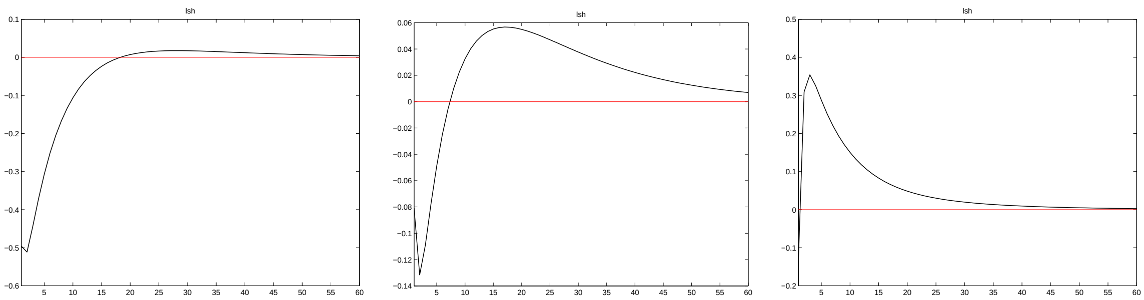
Italics indicate not significantly different from zero using GMM 95% confidence intervals.

Table 8: Theoretical moments using posterior modes: CESCD3 and CESCD4 (technology choice models)

	CESCD3 (permanent Z and X shocks)				CESCD4 (permanent X and Q shocks)			
	Short-run		Medium run		Short-run		Medium run	
	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)	Std(X)/Sdt(Y)	Cor(X,Y)
C	0.730	0.897	0.864	0.986	0.651	0.885	0.867	0.956
Inv	2.222	0.912	1.489	0.960	2.479	0.919	1.704	0.885
W	0.730	0.938	0.893	0.992	0.690	0.825	0.890	0.942
L	0.386	0.262	0.171	0.652	0.298	*-0.081*	0.248	0.130
Prod	0.973	0.924	0.898	0.990	1.066	0.960	0.999	0.969
LSH	0.268	-0.535	0.070	*-0.009*	0.561	-0.547	0.356	-0.211

Italics indicate not significantly different from zero using GMM 95% confidence intervals.

It is also worth mentioning that a model comparison based on posterior odds ratios between the models with choice of techniques and a standard RBC with Cobb-Douglas favour dramatically the former. For instance, comparing CESCD2 with RBC2, if we assign equal prior model probabilities, the posterior probability of model CESCD2 is 0.99996. Similar results are obtained from comparing CESCD3 and CESCD4 with RBC2. When comparing with CES only models, the same picture arises. For instance, the posterior probability of model CESCD3 when compared to CES3 is essentially 1. Thus, with similar subjective priors for any pair of models, the data favor consistently the models with choice of appropriate technology.



(a) Z shock (b) X shock (c) Q shock

Figure 5: Impulse response for the labour share to a 1% Z, X, and Q shocks. $\tau = 20$, $\sigma = 0.2$, $\kappa_i = 0.2$.

Finally, figure 5 presents impulse responses for the labour share to a 1% shock for the three technological processes considered. For this, we fixed τ at a value of 20, and the persistence of the rate of growth shocks to 0.2, which is a common value found in our estimates. We can see that both Z_t and X_t shocks generate a negative response of the labour share, as the short run elasticity of substitution is below one. For Q_t, the response is almost immediately positive, as investment-specific shocks mimic capital-augmenting shocks with a one period delay. An important aspect to highlight is that, especially in the case of Hicks-neutral shocks, the labour share experiences an overshooting after a productivity innovation. The labour share first falls and then increases above its steady state level after around 6-7 quarters, reaches its peak around quarter 15-16, and then returns to equilibrium. Our model is therefore able to reproduce the overshooting of the labour share after a productivity innovation reported in Ríos-Rull and Santaeulàlia-Llopis (2010) using a structural VAR for US data. The effect we find for Hicks-neutral shocks is very close to their estimates. They find that, after an identified productivity shock, the labour share falls for about 5 quarters and then reaches a peak in slightly less than 5 years. This is close to the pattern we find with Hicks-neutral shocks in figure 5.

4 Conclusions

We argue that modelling technology choice presents at least two distinct advantages for macroeconomic modelling. The first is that analysing the shape of a technology frontier arguably provides a richer context for analysing balanced growth and the long-run growth path than simply thinking about the elasticity of substitution between capital and labour. This is because the shape of the frontier determines jointly how the capital share and the long-run elasticity of substitution evolve along the long-run growth path. The paper provides a general theoretical characterisation of this process.

The second advantage is that technology choice naturally leads to a situation where the elasticity of substitution between capital and labour is larger in the long run than in the short run. A particular focus of this framework is to provide a tractable and easily-implementable resolution to the 'balanced growth conundrum' created by the $SSGT$ without requiring explicit models of R&D. If balanced growth is a good description of the long-run growth path, this prevents the inclusion of certain kinds of technical progress in macroeconomomic models as permanent phenomenona if, in accordance with empirical evidence, the elasticity of capital-labour substitution is below one. Using the above framework, we can straightforwardly derive a general production function where the elasticity of capital-labour substitution is less than one in the short run but converges to one in the long run. This leads to a class of production functions that are consistent with balanced growth even in the presence of permanent investment-specific or other kinds of biased technical progress, but where short-run dynamics are characterised by complementarity.

As an application, we present a stochastic general equilibrium business cycle model with the production technology and estimate it using US data for the 1948:Q1-2013:Q3 period. We show that the model does a good job at matching the behavior of the labour share of income at short and medium run frequencies: the labour share is counter-cyclical and volatile in the short run, and almost a-cyclical and smoother in the medium run. The model also performs well in terms of data moments and statistical behavior against a standard RBC model with Cobb-Douglas, and an RBC model with short and long run CES only. It is also capable of reproducing the overshooting of the labour share in reaction to a technology innovation reported in previous studies. Extensions of this approach for further research could consider its introduction in multi-sector growth models, the estimation of technology frontiers, a more detailed specification of the labour market, and a richer set of non-technology shocks.

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Appendix

A Proof of Lemma 1

The technology frontier is weakly bounded if and only if

$$\lim_{\theta\to\infty} a(\theta) = \lim_{\theta\to 0} b(\theta) = -\infty.$$

□

Proof. The technology frontier is defined (definition 1) as weakly bounded iff the sets $R_A \equiv \{A \mid A > 0 \text{ and } \Phi_A(A\mathrm{e}^{-x}) + \Phi_B(B\mathrm{e}^{-x}) \leq 0 \ \forall\ B \geq 0\}$ and $R_B \equiv \{B \mid B > 0 \text{ and } \Phi_A(A\mathrm{e}^{-x}) + \Phi_B(B\mathrm{e}^{-x}) \leq 0 \ \forall\ A \geq 0\}$ are both empty.

Let $C_A = \inf\{A(\theta; x)\ \forall\ \theta \geq 0\}$ and $C_B = \inf\{B(\theta; x)\ \forall\ \theta \geq 0\}$. We can see that R_A is empty iff $C_A = 0$. If $C_A = 0$, for any $\tilde{A} > 0$, there must be some $\tilde{\theta}$ such that $\tilde{A} = A(\tilde{\theta}; x)$, and so for any $\tilde{B} > B(\tilde{\theta}; x)$, $(\tilde{A}, \tilde{B})$ will lie outside the frontier. Thus $\tilde{A}$ cannot belong to R_A, and R_A must be empty. If $C_A > 0$, and $(\tilde{A}, \tilde{B})$ such that $0 < \tilde{A} < C_A$ and $\tilde{B} \geq 0$ must lie within the frontier (consider the ray from the origin through $(\tilde{A}, \tilde{B})$) and so R_A cannot be empty. Similarly R_B is empty iff $C_B = 0$.

Since $a(\theta; x) \equiv \ln A(\theta; x)$ is strictly monotonically decreasing in θ, $\lim_{\theta\to\infty} a(\theta)$ is well-defined and equals $\ln(C_A)$. Thus R_A is empty iff $\lim_{\theta\to\infty} a(\theta) = -\infty$. Similarly, since $b(\theta; x) \equiv B(\theta; x)$ is strictly monotonically increasing in θ, $lim_{\theta\to 0} = C_B$ so R_B
■

B Proof of proposition 2

Part *i)*: Uniqueness

A sufficient condition for equations (12) and (13) to have a unique solution (k^*, θ^*) for all $\Lambda \in \Lambda^S$ is that the frontier has limited convexity everywhere. A necessary condition is the frontier either has limited convexity everywhere or nowhere.
Proof. The first order conditions (12) and (13) are re-written here for convenience.

$$\theta^\rho k^{1-\rho} = \Lambda. \tag{B1}$$

$$\frac{k^\rho}{\theta^\rho} = s(\theta). \tag{B2}$$

We want to establish that (B1) and (B2) have at most one solution for any given Λ. Taking logs and total derivatives of (B2), we get

$$\mathrm{d}\ln\theta = \frac{\rho}{\rho+\eta}\,\mathrm{d}\ln k \tag{B3}$$

and,

$$\frac{1}{\mathrm{dln}\, k}\left(\rho\, \mathrm{dln}\, \theta + (1-\rho)\, \mathrm{dln}\, k\right) = \frac{\rho + (1-\rho)\eta}{\rho + \eta}. \tag{B4}$$

If we have limited convexity everywhere, the left-hand side of equation (B1) can be expressed as a strictly increasing function of k, and therefore equations (B1) and (B2) have a unique solution for k and θ where k must be increasing in Λ. Since $\eta(\theta)$ is continuous in θ, if the frontier has limited convexity on some regions of the frontier and not others, the left-hand side of equation (B1) cannot be strictly monotonic in k, and hence there must exist some factor price ratios for which (12) and (13) have multiple solutions. ■

Part *ii)*: Second order conditions

Any solution (k^*, θ^*) to the first order conditions (10), (11) and (13), and therefore to (12) and (13), will locally maximise the firm's profits if the frontier has limited convexity at θ^*.

Proof. Let us write $Y(K, L, \theta) = T^{\frac{1}{\rho}}$ where

$$T = \mathrm{e}^{\rho a(\theta;x)} K^\rho + \mathrm{e}^{\rho b(\theta;x)} L^\rho = \mathrm{e}^{\rho a(\theta;x)} K^\rho + \theta^\rho \mathrm{e}^{\rho a(\theta;x)} L^\rho. \tag{B5}$$

The first order condition for θ, which gives (B2), is $Y_\theta = {}^1/_\rho Y^{1-\rho} T_\theta = 0$. Hence at any solution to the first order conditions we have $T_\theta = 0$. To prove that such a solution is a local maximum, it is sufficient to show that the Hessian H is negative semi-definite whenever $T_\theta = 0$, where

$$H = \begin{pmatrix} Y_{KK} & Y_{KL} & Y_{K\theta} \\ Y_{KL} & Y_{LL} & Y_{L\theta} \\ Y_{K\theta} & Y_{L\theta} & Y_{\theta\theta} \end{pmatrix}.$$

Note that due to the fact that Y is constant returns to scale in K and L, we have

$$Y_{KK} = \frac{1}{k} Y_{KL}\ ;\ Y_{LL} = k Y_{KL}. \tag{B6}$$

We can also show that whenever $T_\theta = 0$,

$$Y_{L\theta} = k Y_{K\theta}. \tag{B7}$$

Equations (B6) and (B7) imply that $\det H = 0$, so, as in the standard case, there is one zero eigenvalue. The characteristic polynomial $\det(H - \lambda I)$ is then

$$-\lambda^3 + \left[Y_{KK} + k^2 Y_{KK} + Y_{\theta\theta}\right]\lambda^2 - \left\{k^2 Y_{KK}^2 + \left(1+k^2\right)\left(Y_{\theta\theta} Y_{KK} - Y_{K\theta}^2\right)\right\}\lambda. \tag{B8}$$

The two remaining eigenvalues will be strictly negative if and only if the term in the square brackets in (B8) (the sum of the eigenvaules) is strictly negative and the term in the curly bracket (their product) is strictly positive. Since $Y_{KK} < 0$, a sufficient condition for this is

$$Y_{\theta\theta} < \frac{Y_{K\theta}^2}{Y_{KK}}. \tag{B9}$$

To analyse (B9), note the following:

$$Y_K = \mathrm{e}^{\rho a(\theta)} \frac{Y^{1-\rho}}{K^{1-\rho}} \ ; \ Y_{K\theta} = \rho a'(\theta) Y_K \ ; \ Y_{KK} = (1-\rho)\frac{Y_K}{K}\left(-1 + \frac{K^\rho}{T}\right). \tag{B10}$$

Using (B2) and (4), we have

$$Y_{KK} = -(1-\rho)\frac{Y_K}{K}\frac{1}{1+s(\theta)} = \frac{1-\rho}{\rho}\frac{\theta Y_{K\theta}}{K}. \tag{B11}$$

Since $T_\theta = \rho \mathrm{e}^{\rho a(\theta)}\{a'(\theta)K^\rho + \theta^\rho b'(\theta) L^\rho\} = 0$,

$$T_{\theta\theta} = \rho \mathrm{e}^{\rho a(\theta)}\{a''(\theta)K^\rho + [\theta^\rho b''(\theta) + \rho\theta^{\rho-1}b'(\theta)]L^\rho\} \tag{B12}$$

and, again using (B2),

$$\begin{aligned} Y_{\theta\theta} &= \frac{1}{\rho}Y^{1-\rho}T_{\theta\theta} = \frac{K^{1-\rho}T_{\theta\theta}}{\rho^2 a'(\theta)\mathrm{e}^{\rho a(\theta)}} Y_{K\theta} \\ &= \frac{K}{\theta\rho}\left(\frac{\theta a''(\theta)}{a'(\theta)} - \frac{\theta b''(\theta)}{b'(\theta)} - \rho\right) Y_{K\theta} = -\frac{K}{\theta\rho}\left(\eta(\theta) + \rho\right) Y_{K\theta}, \end{aligned} \tag{B13}$$

noting that $\eta(\theta) = {}^{\theta b''(\theta)}/_{b'(\theta)} - {}^{\theta a''(\theta)}/_{a'(\theta)}$. Equations (B11) and (B13) then imply that condition (B9) is satisfied if and only if

$$\eta(\theta) < \frac{-\rho}{1-\rho}. \tag{B14}$$

Hence limited convexity implies that the Hessian is negative semi-definite, and hence any solution to the first-order conditions is a local maximum.

■

Part *iii)*: Corner Solutions

Suppose $\rho < 0$, the frontier has limited convexity and the frontier is weakly bounded. If (12) and (13) have a solution (k^*, θ^*), then it is unique (by part *i)*) and $(k^*, \theta^*, r + \delta, w)$ is the unique equilibrium and an interior one.

Proof. Given factor prices $r + \delta$ and w, limited convexity implies (by part *i)* of the proposition) that the solution (k^*, θ^*) to equations (12) and (13) is unique. Therefore there can be no more than one interior equilibrium.

Suppose a firm chooses the triplet (K, L, θ) with capital-labour ratio $k = {}^{K}\!/_{L}$. This is an *interior choice* if both $k \in (0, \infty)$ and $\theta \in (0, \infty)$, and a *corner choice* otherwise. When $\rho < 0$, a weakly bounded frontier straightforwardly implies that a corner choice implies zero output. Thus no corner equilibrium can exist. Thus to complete the proof we need to show that $(k^*, \theta^*, r + \delta, w)$ is an interior equilibrium, i.e. that if all firms choose (k^*, θ^*), no firm has an incentive to deviate. This follows immediately since a firm that deviates has no profit-maximising choice. Suppose it has. If it is an interior choice it must satisfy the first order conditions (10), (11) and (13), and therefore also (12). This cannot be the case however since (12) and (13) have a unique solution. However, it cannot be a corner choice, since that will yield zero output. Therefore the proposition follows. ■

C Proof of proposition 5

In problem $P1$, firms choose K and L to maximise $Y - (r + \delta)K - wL$ given factor prices $r + \delta$ and w and a CES production technology with elasticity of substitution $\sigma_{LR} = \frac{1}{1-\mathcal{R}}$ with $\mathcal{R} < 1$:

$$Y = \begin{cases} X\left(\alpha K^{\mathcal{R}} + (1-\alpha)L^{\mathcal{R}}\right)^{\frac{1}{\mathcal{R}}} & \text{when } \mathcal{R} \neq 0 \\ XK^{\alpha}L^{1-\alpha} & \text{when } \mathcal{R} = 0. \end{cases} \tag{C1}$$

In problem P_2, firms choose K, L and θ to maximise $Y - (r + \delta)K - wL$ where, writing $x = \ln X$, Y is given by

$$Y = \left[\mathrm{e}^{\rho a(\theta;x)}K^{\rho} + \theta^{\rho}\mathrm{e}^{\rho a(\theta;x)}L^{\rho}\right]^{1/\rho}, \tag{C2}$$

where we assume $\mathcal{R} > \rho$. Proposition 5 and its corollary then state:

If P_2 has an interior equilibrium solution (which is therefore described by equations (12) and (13) above) this solution will result in identical outcomes for $y \equiv {}^{Y}\!/_{L}$ and $k \equiv {}^{K}\!/_{L}$ to problem P_1 if and only if the function $a(\theta; x)$ takes the form

$$a(\theta; x, \mathcal{R}) = \begin{cases} x + \frac{1}{\mathcal{R}\zeta}\ln\left(\alpha^{\zeta} + (1-\alpha)^{\zeta}\theta^{-\mathcal{R}\zeta}\right) & \text{when } \mathcal{R} \neq 0 \\ x + \frac{1}{\rho}\left[\alpha\ln\alpha + (1-\alpha)\ln(1-\alpha)\right] - (1-\alpha)\ln\theta & \text{when } \mathcal{R} = 0 \end{cases} \tag{C3}$$

where the constant $\zeta \equiv \frac{\rho}{\rho - \mathcal{R}}$. Noting that $\zeta \to 1$ as $\mathcal{R} \to 0$ and that $\frac{\partial\zeta}{\partial\mathcal{R}}|_{\mathcal{R}=0} = \frac{1}{\rho}$, it follows by L'Hôpital's rule that $a(\theta; x, \mathcal{R})$ is continuous in $\mathcal{R}$. The functions for the slope and elasticity of the frontier implied by (21) are:

$$s(\theta; \mathcal{R}) = \left(\frac{\alpha}{1-\alpha}\right)^{\zeta}\theta^{\mathcal{R}\zeta}\ ;\ \eta(\theta; \mathcal{R}) = \mathcal{R}\zeta < \frac{-\rho}{1-\rho}. \tag{C4}$$

It follows that:

i) If $\rho\mathcal{R} > 0$, then the frontier (21) is both weakly and strongly bounded; if $\mathcal{R} = 0$, then it is weakly but not strongly bounded; if $\rho\mathcal{R} < 0$ then it is not weakly (and therefore not strongly) bounded.

ii) If $\mathcal{R} \leq 0$, since the frontier (21) is weakly bounded and has limited convexity everywhere, then, by proposition 2, P_2 has a unique equilibrium which is an interior one.

iii) If $\mathcal{R} > 0$ and conditions (18) and (19) hold, then the unique solution to equations (12) and (13) describes an interior equilibrium for P_2.

Proof. For problem P_1, we obtain

$$\frac{1-\alpha}{\alpha}k^{1-\mathcal{R}} = \Lambda. \tag{C5}$$

If problem $P2$ has an interior solution, this solution is described by equations (12) and (13), repeated here for convenience:

$$\theta^{\rho}k^{1-\rho} = \Lambda, \tag{C6}$$

$$\frac{k^{\rho}}{\theta^{\rho}} = s(\theta). \tag{C7}$$

Equations (C5), (C6) and (C7) must have the same solution for k for any factor price ration Λ. Hence we must have

$$\theta^{\rho} = \frac{1-\alpha}{\alpha}k^{\rho-\mathcal{R}} \tag{C8}$$

and

$$s(\theta) = -\frac{1}{\theta a'(\theta)} - 1 = \left(\frac{\alpha}{1-\alpha}\right)^{\zeta}\theta^{\zeta\mathcal{R}}. \tag{C9}$$

where $\zeta \equiv \frac{\rho}{\rho-\mathcal{R}}$. The differential equation (C9) has the following solution for $a(\theta)$

$$a(\theta;\mathcal{R}) = \begin{cases} c_1 + \frac{1}{\mathcal{R}\zeta}\ln\left(\alpha^{\zeta} + (1-\alpha)^{\zeta}\theta^{-\mathcal{R}\zeta}\right) & \text{when } \mathcal{R} \neq 0 \\ c_2 - (1-\alpha)\ln\theta & \text{when } \mathcal{R} = 0 \end{cases} \tag{C10}$$

For problem P_1, writing $y = {}^{Y}\!/_{L}$, (C1) implies

$$y = \begin{cases} X\left(\alpha k^{\mathcal{R}} + (1-\alpha)\right)^{\frac{1}{\mathcal{R}}} & \text{when } \mathcal{R} \neq 0 \\ Xk^{\alpha} & \text{when } \mathcal{R} = 0. \end{cases} \tag{C11}$$

Substituting from (C10) and (C7) into equation (C2) should give us expressions for y equivalent to (C11). This enables us to extract the constants c_1 and c_2:

$$c_1 = \ln X = x \tag{C12}$$
$$c_2 = x + \frac{1}{\rho}\left[\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right].$$

The shape of the frontier specified by (C10) and (C12) is then both necessary and sufficient to ensure that P_1 and P_2 have identical outcomes for y and k. The function for $\eta(\theta;\mathcal{R})$ given in the proposition follows immediately from (C9).

Corollary: For part i) of the corollary, note that

$$b(\theta;\mathcal{R}) = \begin{cases} x + \frac{1}{\mathcal{R}\zeta}\ln\left(\alpha^{\zeta}\theta^{\mathcal{R}\zeta} + (1-\alpha)^{\zeta}\right) & \text{when } \mathcal{R} \neq 0 \\ x + \frac{1}{\rho}\left[\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\right] + \alpha \ln \theta & \text{when } \mathcal{R} = 0 \end{cases} \tag{C13}$$

When $\mathcal{R} = 0$ the frontier is clearly weakly but not strongly bounded. Since we have $\rho - \mathcal{R} < 0$ the sign of $\mathcal{R}\zeta$ depends only on the sign of $\rho\mathcal{R}$. When $\rho\mathcal{R} > 0$, $\mathcal{R}\zeta < 0$ and so both $a(\theta)$ and $b(\theta)$ are bounded above, so the frontier is both weakly and strongly bounded. When $\rho\mathcal{R} < 0$, $\mathcal{R}\zeta > 0$ and so $\lim_{\theta\to\infty} b(\theta) = \lim_{\theta\to 0} a(\theta) = \infty$, so the frontier is not strongly bounded. It is also not weakly bounded; this follows from lemma 1. Part ii) of the corollary follows directly from part i) and proposition 2.

For part iii), where $\mathcal{R} > 0$, we need to show that $(k^*, \theta^*, r+\delta, w)$ is an interior equilibrium where k^* and θ^* are the unique solutions to (12) and (13). That is, if all firms chose (k^*, θ^*) (and so make zero profits) no firm has an incentive to deviate. Following the proof of part iii) of proposition 2, due to the uniqueness of the solution to the first order conditions, it sufficient to show that any corner choice must decrease profits.

Note that if a firm makes a corner choice in k it will be optimal to make a corner choice in θ, so any corner choice must have the property that $\theta = 0$ or $\theta = \infty$. Using (C3) and (C13), when $\rho < 0$ and $\theta = \infty$, or when $\rho > 0$ and $\theta = 0$, the production function takes the form

$$Y = X\alpha^{\frac{1}{\mathcal{R}}}K. \tag{C14}$$

Similarly when $\rho > 0$ and $\theta = \infty$, or when $\rho < 0$ and $\theta = 0$, the production function takes the form

$$Y = X(1-\alpha)^{\frac{1}{\mathcal{R}}}L. \tag{C15}$$

Thus if conditions (18) and (19) hold, then a corner choice must yield negative profits.

■

D The user cost of capital and the capital share

Since the production technology developed in the main paper is Cobb-Douglas in the long run, there is no long-run relationship between the capital share and the real user cost of capital. Thus a trend in the real user cost of capital – due, for example, to changes in depreciation rates – would cause no trend in the capital income share. Because the production function is CES in the short run, from the marginal product of capital condition, the short-run elasticity between these two quantities will be $1-\sigma$. Suppose instead we had a standard CES production function in both the short and long run, with purely labour augmenting progress for compatibility with balanced growth. In that case, in addition to the short-run correlation, we would expect a long-run co-integrating relationship between the log of the capital share and the log of the real user cost with an elasticity $1-\sigma$.

Figure 6 plots the joint evolution of the log of the capital income share of the non-farm private business sector and the log of the real user cost for the US for the 1952:Q1-2004:Q4 period.[24] The capital income share is calculated as a residual after deducting wages and imputed self-employed income from the private sector GDP. The real user cost is simply the ratio between this imputed capital income and the capital stock.

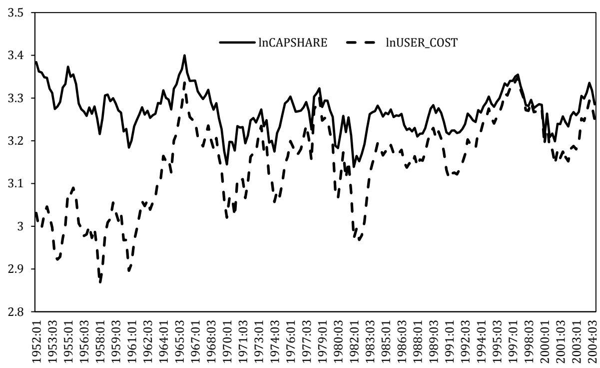

Figure 6: Log of the capital income share (solid) and log of the real user cost (dashed)

A close correlation between the two series can be observed in the short run. Using a Band Pass filter to separate trend and cyclical components, the correlation coefficient between the cyclical components is 0.91. Regressing the cyclical components yields an estimated σ of 0.42. In the long-run, however, there is a sizeable departure between the two series, and the two series are clearly not co-integrated. Though the short run correlation is consistent with a CES production function, the long-run pattern is more akin to the one arising from a Cobb-Douglas. Clearly, this is far short of making this empirical distinction a rigorous one, but it provides evidence that the short/long-run properties of the production technology may at least be consistent with the data.

[24]We used data from Klump et al (2007).

E Data sources and construction

E.1 The labour share

Measuring the share of labour in total income is complicated by problems associated with how to impute certain categories of income to labour and capital owners. The existence of self-employment income, the treatment of the government sector, the role of indirect taxes and subsidies, household income accruing from owner occupied housing, and the treatment of capital depreciation, are common problems highlighted in the literature. These have been discussed at length in Gollin (2002) and Gomme and Rupert (2004). In constructing the labour share data for the US, where data on income sources is richest, we use three different measures. The first is directly taken from the BLS, and the other two are based on Gomme and Rupert (2004). For other countries, where available, we will use similar measures. However, data availability limits the extent to which we can obtain corrected labour share measures and, in many cases, we work with rough estimates of labour shares.

E.1.1 US labour share

The three measures used for the US are constructed using data from the BLS and the BEA NIPA Tables and are as follows:

1. *Labour share 1*: Labour share in the non-farm business sector. This is taken directly from BLS.[25] The series considers only the non-farm business sector. It calculates the labour share as compensation of employees of the non-farm business sector plus imputed self-employment income over gross value added of the non-farm business sector. Self-employment imputed income is calculated as follows: an implicit wage is calculated as compensation over hours worked and then the imputed labour income is the implicit wage times the number of hours worked by the self-employed.

2. *Labour share 2*: Labour share in the domestic corporate non-financial business sector. This follows Gomme and Rupert's (2004) first alternative measure of the labour share. The use of data for the non-financial corporate sector only has the advantage of not having to apportion proprietors income and rental income, two ambiguous components of factor income. It also considers the wedge introduced between the labour share and one minus the capital share by indirect taxes (net of subsidies), and only makes use of unambiguous components of capital income. The labour share is thus calculated as:

$$\text{LSH2} = 1 - \frac{\text{CORPORATE PROFITS + NET INTEREST - NET IND. TAXES}}{\text{NET VALUE ADDED}}.$$

[25] FRED series PRS85006173 provided as an index number.

3. *Labour share 3*: imputing ambiguous income for the macroeconomy. This corresponds to the second alternative measure of the labour share proposed in Gomme and Rupert (2004). The measure excludes the household and government sectors. They define unambiguous labour income (Y^{UL}) as compensation of employees, and unambiguous capital income (Y^{UK}) as corporate profits, rental income, net interest income, and depreciation. The remaining (ambiguous) components are then proprietors' income plus indirect taxes net of subsidies. These are apportioned to capital and labour in the same proportion as the unambiguous components. The resulting labour share measure is:

$$\text{LSH3} = \frac{Y^{UL}}{Y^{UK} + Y^{UL}}.$$

E.1.2 Other countries

We constructed measures of the labour share on a quarterly basis for some countries for which data were available for a sufficiently long period of time. In the descriptive analysis, we used data for Australia, Canada, The Netherlands, Spain, and the UK. For most of these countries, the level of detail of the income accounts did not allow us to construct measures consistent with those of the US. In what follows, we describe the different measures used for these countries:

1. *Australia.* Quarterly data for the 1959:Q1-2013:Q3 from the Australian Bureau of Statistics. One minus gross operating surplus of private non-financial corporations plus all financial corporations as a percentage of total factor income.[26]

2. *Canada.* Quarterly data for the 1981:Q1-2013:Q3 period from Statistics Canada. Compensation of employees over total factor income. [27]

3. *The Netherlands.* Quarterly data for the 1988:Q1-2013:Q3 from the Central Bureau of Statistics. The longest time series available allowed us to construct the series for the labour share as one minus gross operating surplus over GDP net of indirect taxes less subsidies.[28]

4. *Spain.* Quarterly data for the 1995:Q1-2011:Q2 period from the National Institute for Statistics. We used compensation of employees over GDP.[29]

[26] Web link: `http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/5206.0Sep%202013?OpenDocument`, Table 7.

[27] Web link: `http://www.statcan.gc.ca/nea-cen/hr2012-rh2012/data-donnees/cansim/tables-tableaux/iea-crd/c380-0063-eng.htm`.

[28] Web link: `http://statline.cbs.nl/StatWeb/selection/?DM=SLEN&PA=81117ENG&LA=EN&VW=T`.

[29] Web link: `http://www.ine.es/jaxiBD/tabla.do?per=03&type=db&divi=CNTR&idtab=10`.

5. *UK*. Quarterly data for the 1955:Q1-2013:Q3 period from the Office for National Statistics. We used compensation of employees over gross value added at factor costs.[30]

E.2 Other US macroeconomic aggregates

Other macroeconomic aggregates were constructed following standard convention in the calibrated business cycle literature. All data were obtained from either the FRED database or directly at the Bureau of Economic Analysis (indicated in parentheses). A brief description follows:

- Output: output in the non-farm business sector over civilian non-institutionalized population (FRED).
- Consumption: real non-durable and services consumption over civilian non-institutionalized population (BEA).
- Investment: real private fixed investment plus durable consumption over civilian non-institutionalized population (BEA).
- Wages: compensation per hour in the non-farm business sector (FRED).
- Hours: all hours in the non-farm business sector over civilian non-institutionalized population (FRED).
- Productivity: output per hour in the non-farm business sector (FRED).
- Relative price of investment: price deflator for durables and investment relative to the deflator for non-durables and services (BEA).
- Civilian non-institutionalized population over 16 from the FRED database.

[30] Web link: `http://www.ons.gov.uk/ons/datasets-and-tables/data-selector.html?table-id=D&dataset=qna`.

Non-technical summary

In macroeconomics, we typically model production by specifying a 'production function,' which tells us how much output is produced with given quantities of the 'factors of production,' often taken simply as capital and labour. Factor shares refer to the proportion of the income earned by production that goes to each factor, so the labour share is the proportion of this income that is earned by workers through supplying labour. There are various issues with how we measure factor shares, but a key aspect of what is known as 'balanced growth' is the idea that as income grows over long periods of time, the labour share remains approximately constant.

Some researchers dispute the idea of balanced growth, arguing for example that the labour share is currently declining. What there is no disagreement about is the fact that these factor shares are far more stable in the long run than they are in the short run. This creates a problem in the way we specify production functions. For example, the assumption of balanced growth has led Cobb-Douglas production functions to become standard in macroeconomic models because they imply constant factor shares with perfectly competitive markets. This, however, makes it more difficult to capture short- and medium-run fluctuations in factor shares. Market failures such as wage and price rigidities allow us to explain some of these fluctuations, but it is unlikely that they account for all of the fluctuations we see in factor shares, particularly in the medium run.

When the relative price of capital (to labour) rises, firms hire relatively less capital and more labour. The elasticity of substitution between capital and labour quantifies this effect; it tells us by how many percent the capital-labour ratio declines when the relative price of capital goes up by 1%. With Cobb-Douglas production functions, this elasticity is always one. However, much of the empirical evidence finds support for an elasticity below one. Indeed, production functions with an elasticity below 1 typically capture short-run fluctuations in factor shares significantly better than Cobb-Douglas. However, they have very important long-run consequences for income distribution. If the elasticity is different to one, productivity changes can cause the labour share to change. Since we have observed permanent changes in the productivity of investment goods in the last 30 years, an elasticity below one would lead to unbalanced growth with an increasing labour share, whereas typically researchers think that it is either constant or declining.

In this paper we propose a solution to this problem, using the idea of "appropriate technology." This is the idea that firms not only choose the quantities of capital and labour to employ, but also make a technology choice – how labour- or capital-intensive they want production methods to be. This trade-off is expressed graphically by a technology frontier: technologies that are more efficient in using labour are less efficient in using capital and vice-versa. Given a change in factor prices, firms change their position on the frontier. We show how the shape of the frontier determines the long-run elasticity of substitution and long-run factor shares. Importantly, if firms face adjustment costs when changing their choice of technology, the short-run elasticity will be lower than the long-run elasticity. This provides a way of modelling production that is very easy to implement in macroeconomic models but that is flexible enough to be compatible with both short- and long-run data. The short-run elasticity can be calibrated to capture short-run fluctuations in factor shares in line with the evidence, while the shape of the frontier captures the properties of long-run growth. There is a specific shape of frontier that implies balanced growth. Here elasticity of substitution is below one in the short-run but adjusts towards one in the long run. We use this to provide a quantitative example for the US economy. The results support the use of this new production function because it improves the model's ability to explain the business cycle and medium-run behaviour of the labour share.