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The Choice of CES Production Techniques and Balanced Growth

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Abstract

We show that allowing firms a choice of CES production techniques (via the distribution parameter between capital and labor) can result in a new class of production functions that produces short-run capital-labor complementarity but yields a long-run unit elasticity of substitution. This is shown to occur if we provide a mathematical framework for this choice that maintains strict essentiality of the production process and satisfies the requirement of unit-invariance. The class of production functions derived are consistent with a balanced growth path even in the presence of capital-augmenting technical progress. The approach yields a simple yet powerful way of introducing CES-type production functions in macroeconomic models.
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1 Introduction

We propose a framework for allowing firms the choice of the distribution parameter between capital and labor, as well as the quantities of capital and labor themselves, in a Constant Elasticity of Substitution (CES)-type production function. This results in a class of production functions with some very convenient properties for the existence of a balanced growth path: they have a long-run unitary elasticity of substitution between capital and labor but a short-run elasticity of substitution less than one if firms face adjustment costs in altering the technological reliance on capital. This class of production functions is then consistent both with traditional stylized facts regarding long-run growth (balanced growth, non-trended real interest rates and non-trended shares of capital and labor), and with mounting evidence that short-run dynamics are better matched by an elasticity of substitution of less than one, while not imposing any a-priori restrictions on the nature of technical progress.

This provides theoretical support to the researcher who wishes to model short- to medium-run dynamics with elasticity of substitution less than one rather than using the much more common Cobb-Douglas assumption, complementing the contribution of Jones (2005) and others which we discuss in more detail below, since this usage is shown not to necessarily contradict stylized facts regarding long-run growth. From the point of view of tractability, however, the approach taken here is arguably much easier to implement than others proposed in the literature such as Houthakker (1955-56) and Jones (2005) in a conventional dynamic framework. We also aim to be of practical use to researchers who wish to consider analyzing the impact of *permanent* shocks to capital augmenting productivity, particularly in terms of their short-run effects. Here, both temporary and permanent capital and labor-augmenting shocks are distinguished by their short-run effects, but any distribution of these shocks is consistent with long-run balanced growth.

This class of production functions takes the following form:

$$Y = \psi(K, L, \alpha) \tag{1}$$

where, in addition to choosing the inputs K and L, the firm has a technological choice regarding α, the distribution or share parameter between K and L. We will use the phrase 'choice of technique' in the paper strictly to refer to the choice of α. Most of the paper concerns the motivation of the appropriate form for $\psi(.)$. We start with the assumption that, treating α as fixed, ψ takes the form of a standard CES production function with an elasticity below one. Given that firms will of course make a profit-maximizing choice of α, we determine the form of $\psi(.)$ using two considerations: (i) strict essentiality, that is, that a strictly positive quantity of each input is required for strictly positive production; and (ii) unit-invariance, that the form of $\psi(K, L, \alpha)$ is invariant to a change of measurement units in K

or L. In fact, these two considerations – or requirements – allow us to pin down a unique class of functional forms for $\psi(K, L, \alpha)$. We then use this to develop a production technology with the properties stated above by introducing adjustment costs in α.[1] Hence the paper can be seen as providing a generalization of CES functions that (i) makes it consistent with balanced growth even in the presence of capital-augmenting technical progress (ii) as an additional convenience, due to the unit invariance property, is free from the normalisation considerations often required by the implementation of CES (see La Grandville 1989, Klump and La Grandville 2000, León-Ledesma *et al* 2010, and Cantore and Levine, 2010). The approach is also very simple and easily implementable in fully fledged dynamic general equilibrium models.

The important drawback in assuming Cobb-Douglas is that it sits at odds with the observed large (short- to medium-run) variations in factor shares, and the available empirical evidence on the elasticity of capital-labor substitution.[2] Clearly, if the elasticity of substitution is incorrectly assumed to be unity in the short-run, the implications for modeling short-run dynamics can be substantial (see Cantore *et al*, 2010). As a result, the Steady State Growth Theorem which states that for a balanced growth path (BGP) to exist either technical progress must be labor-augmenting or the production function must be Cobb-Douglas (Uzawa, 1961) *in the long run*,[3] becomes potentially restrictive if one wants to satisfy a BGP and still accurately model short-run dynamics.

A related important stream of the literature therefore concerns "induced innovation", which provides an explanation of why technical progress may all be labor-augmenting in the long-run. The early literature on induced innovation by Kennedy (1964), Samuelson (1965), and Drandakis and Phelps (1966), inspired by Hicks (1932), viewed this as the result of firms introducing innovations that save on expensive factors in the face of changes in relative factor prices. More recently, this line of thought has been revisited by Acemoglu (2002, 2003, 2007) and Zeira (1998) amongst others. An adequate survey is far beyond the scope of this paper, as is the question of whether the induced innovation literature as a whole produces the outcome of balanced growth without overly-restrictive assumptions on the nature of innovation.[4] At the very least, to address the problem of balanced growth, the induced innovation literature requires a formal modeling of innovation. This can potentially make departures from the standard Cobb-Douglas framework difficult when the research question does not concern innovation itself or in situations where it is convenient to treat technical progress as exogenous.

[1]More precisely, as will be clear below, we introduce adjustment costs in $\theta \equiv (1 - \alpha)/\alpha$.

[2]Indeed, the weight of evidence reviewed in, e.g., Chirinko (2008) and León-Ledesma *et al* (2010), supports a value of this elasticity well below unity.

[3]See also Jones and Scrimgeour (2008) for a proof.

[4]See Acemoglu (2003) for a very useful discussion.

A far smaller literature – recently exemplified by Jones (2005) and papers that follow – relies on exploiting the fact that all that is requited for BGP is that the production function be Cobb-Douglas 'in the long run.' Jones (2005), as we do here, provides a production frontier that is Cobb-Douglas in the long-run but where the elasticity of substitution falls short of unity in the short-run. While our approach is complementary to Jones (2005)– for instance both approaches produce Cobb-Douglas at the firm level, rather than as a result of aggregation,[5] and come from a choice of technology by firms – it is perhaps worthwhile noting some key differences. The most important one is in the approach that is taken to justify the functional forms that result in long-run Cobb-Douglas. In Jones (2005) it comes through an arrival process for ideas: it depends on (and is supported by evidence) that this process is governed by Pareto distributions. In our case, we rely on a theoretical justification by seeking to generalize CES functions in a plausible way to provide *unit-invariance*. While the link between an arrival process for ideas and a long-run production function is clearly a very attractive one, perhaps a key advantage of the approach taken here is tractability; the asymptotic convergence to a production frontier and the consequent extreme value nature of the problem are replaced here by a straightforward production technology and just one extra first order condition.[6] Finally, Jones (2005) produces the result that technical progress is labor augmenting, whereas here technical progress is exogenous and labor or capital–augmenting.[7]

The rest of the paper is organized as follows. Next section presents the production technology, discusses unit-invariance, and the dynamics of the model. Section 3 concludes.

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[5]The aggregation approach is taken by Houthakker (1995-56). Jones (2005) and Lagos (2006) provide a very useful discussion of this classic paper. Lagos (2006), in the spirit of Houthakker, derives a Cobb-Douglas form for the aggregate production function by aggregating Leontief production technologies at the firm level using a model with search frictions. The aims of that paper are very different to what is discussed here (for instance it assumes an exogenous rental on capital) and are primarily directed at accounting for the determinants of observed TFP. We aim here to provide a production technology that can be implemented in a wide class of macroeconomic models.

[6]This is also likely to have a significant impact on the short-run dynamics of factor shares. In Jones, due to the extreme value nature again, these factor shares can be quite volatile. Here, a standard adjustment cost mechanism results in smooth movements in factor shares. However, judging between these two results empirically is left for future work.

[7]Note that Jones (2005) also concludes that "[...] Alternatively, it might be desirable to have microfoundations for a Cobb-Douglas production function that permits capital-augmenting technological change to occur in the steady state." Since the long run is Cobb-Douglas, labor- and capital-augmenting shocks are only distinguished in the short-run of course.

2 The production technology

2.1 CES and Capital Intensity

Take the CES production function, omitting any time subscripts:

$$Y = \Gamma \left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where K is capital, L labor, B and A are capital- and labor-augmenting technical progress functions, Γ is a neutral efficiency parameter, σ is the elasticity of substitution between K and L, and α is the capital intensity of production. Suppose now firms are faced with (2) and factor prices $r + \delta$ and w, but now have the additional task of choosing α as well as the inputs K and L, and we have a standard free entry condition. Clearly, from (2) we will not have a satisfactory second order condition for α. We are, in fact, about to deem the outcome of this problem (described in lemma 1) as 'unrealistic' and reformulate it. However it is useful for further discussion and the interesting point is that even though all firms are identical, the outcome must be asymmetric.

Lemma 1 *Some firms will only employ capital and others will only employ labor. No firm will employ both.*

Proof. Suppose $r + \delta < \Gamma B$. Then a firm entering the market choosing $\alpha = 1$ will make a strictly positive profit $(\Gamma B - r - \delta)K$ violating the free-entry condition. Similarly if $w < \Gamma A$ firms can enter and make strictly positive profits choosing $\alpha = 0$. Thus in equilibrium the factor prices must be $r + \delta = \Gamma B$ and $w = \Gamma A$. Thus a firm choosing $\alpha \in (0, 1)$ will earn a profit of Γ times

$$\left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - BK - AL.$$

But $\left(\alpha (BK)^{(\sigma-1)/\sigma} + (1-\alpha)(AL)^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} \leq \max(BK, AL)$ and if both K and L are strictly positive, $\max(BK, AL) < BK + AL$. Hence no firm that employs both factors can make positive profits. ■

Much of our approach is concerned with how to modify the problem of choosing α in order to end up with a more sensible and symmetric solution that can be described as the outcome of an optimizing decision by a representative firm. This modification turns out to produce a very simple framework that also has some very convenient properties for balanced growth. We make two changes. These are partly aimed in the first instance at ensuring strict essentiality of the production function in order to rule out the type of asymmetric equilibrium of lemma 1, and ensuring that an appropriate second order condition is satisfied for α. The CES production function does not satisfy strict essentiality when $\sigma > 1$. This does not normally matter since the marginal product of each factor tends to infinity at zero, but when α is a choice variable the possibility of one firm specializing in one factor and bidding up the price of that factor becomes a real one, which means an asymmetric equilibrium is possible.[8]

The first change is therefore to impose $\sigma < 1$, though this only ensures strict essentiality for CES if $\alpha \in (0, 1)$. The second more substantial change is to introduce a term $f(\alpha)$ into the production function as shown in (3):

$$Y = \Gamma f(\alpha) \left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} . \tag{3}$$

The introduction of $f(\alpha)$, which is continuously differentiable and positive on $\alpha \in [0, 1]$, captures the idea that extreme choices of α are likely to be costly since they imply all of the tasks being performed by capital or labor. If there is comparative advantage, a more balanced distribution of tasks between capital and labor could be more "efficiency enhancing."[9] Technically, it is easy to see that an appropriate form for the function $f(\alpha)$ can result in a well-defined problem with an interior solution for α with a suitably satisfied second order condition. The obvious danger of this approach is that the form of $f(\alpha)$ is potentially rather arbitrary. Before going any further, we discuss transformations of the CES under a change of units. This turns out to yield several rewards: (i) we argue that it removes the arbitrariness surrounding the choice of $f(\alpha)$; (ii) introducing $f(\alpha)$ provides a way of surmounting the various issues that have been raised in the literature regarding changing units within CES production functions; and (iii) the form that arises for $f(\alpha)$ through considering unit-invariance turns out to have very convenient properties for balanced growth. The latter point arises because achieving unit invariance results in a long-run production function that is Cobb-Douglas.

2.2 Measurement units in CES

We repeat the CES production function for convenience:

$$Y = \Gamma \left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} . \tag{4}$$

In an ideal world, the terms BK and AL would have the same units as output. In such a world, the question of units is essentially irrelevant: a doubling of L

[8]One could fairly dismiss this as an unrealistic 'nuisance' equilibrium. However in situations where the elasticity of substitution between two factors might be above 1, skilled and unskilled labor for example, allowing firms to choose might explain specialisation among firms in one factor or the other without relying on any ex-ante heterogeneity.

[9]We are grateful to Daron Acemoglu for this interpretation.

would simply be met by a halving of A. While this is by far the most theoretically appealing interpretation of the CES function, no methodology has been developed in the literature to implement this interpretation when confronted with the data. Given L, measured say in hours, by what quantity A should we multiply L to end up with a quantity that can be measured in the same units as output? Hence, in most practical applications the terms AL and BK cannot in general, with any certainty, be considered to have the same units as output. As a result, these terms can be scaled in a somewhat arbitrary way, and researchers have therefore addressed the question of what are the consequences of scaling AL differently. In particular, in the absence of a method for scaling AL in relation to units of output, the normalisation literature on CES production functions essentially provides a methodology for scaling terms such as AL in units relative to 'baseline' values[10] (e.g. see La Grandville, 1989, 2008, Klump and La Grandville, 2000, Klump and Saam, 2008, and León-Ledesma *et al*, 2010). As is well known, of the three parameters α, Γ, and σ, only σ is invariant to the choice of units. For example, suppose we maintain the same units for Y and K but use a different measure of labor $L' = s^{\frac{\sigma}{1-\sigma}} L$ where s is some scaling constant. Then equation (2) becomes:

$$Y = \Gamma' \left(\alpha' (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha')(AL')^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{5}$$

where

$$\alpha' = \frac{\alpha}{\alpha + (1-\alpha)s} \text{ and } \Gamma' = \Gamma \left(\alpha + (1-\alpha)s\right)^{\frac{\sigma}{\sigma-1}}. \tag{6}$$

Equations (5) and (6) simply describe a transformation that keeps Y and K the same while maintaining 'shares' that add up to 1 within CES.[11] A somewhat similar transformation is proposed in Cantore and Levine (2010) in the context of calibrated macro models.

2.3 Unit Invariance with $f(\alpha)$

Consider again the expression for Γ' given in (6): due to the change of units of labor, an expression that depends on α arises outside the main CES bracket. If we therefore adopted a formulation such as (3), in general we would expect the *functional form* for $f(\alpha)$ to change with a change in units. Clearly this would not be a satisfactory situation. In light of this, we face the requirement to make a consistent representation of the production function in the face of a change of

[10] Allowing the expression of the CES production function in index form.

[11] Consider the marginal product of capital under this unit change. It is $\alpha(B\Gamma)^{\frac{\sigma-1}{\sigma}} \left(\frac{Y}{K}\right)^{\frac{1}{\sigma}}$. Since the change of units of labor leaves Y and K unchanged, and $\alpha\Gamma^{\frac{\sigma-1}{\sigma}} = \alpha'\Gamma'^{\frac{\sigma-1}{\sigma}}$ the marginal product of capital is left unchanged, as it should be, by a change of units to labor.

units. Given the introduction of $f(\alpha)$, this is the key step that yields our main results. Suppose we consider the same change of units above where we use a different measure of labor $L' = s^{\frac{\sigma}{1-\sigma}}L$ where s is constant. We wish to write:

$$Y = \Gamma' \left[f\left(\alpha'\right) \left(\alpha'(BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha')(AL')^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \right], \tag{7}$$

where, as before, we must have $\alpha' = \frac{\alpha}{\alpha+(1-\alpha)s}$.

If (3) is to be a consistent representation of the production function in the face of a change of units, $f(.)$ *must* capture the *whole* of the dependency of Y on α' outside the standard CES term in (7); so in (7) we do not want any term in α' outside the square brackets. Equivalently, we require the functional form for $f(.)$ to stay invariant following the change of units. Hence Γ' cannot be a function of α' (or therefore α) and can only depend on Γ, s, and σ. Suppose $\Gamma' = \phi(s;\sigma)\Gamma$ for some function $\phi(s;\sigma)$. We must then have:

$$\frac{f(\alpha)}{(\alpha+(1-\alpha)s)^{\frac{\sigma}{1-\sigma}}} = f\left(\alpha'\right)\phi(s) = f\left(\frac{\alpha}{\alpha+(1-\alpha)s}\right)\phi(s). \tag{8}$$

We can then show:

Lemma 2 *Suppose for any α and s on the intervals $\alpha \in [0,1]$ and $s \in (0,\infty)$, there exists a function $\phi(s)$ independent of α such that a continuously differentiable function $f(\alpha)$ satisfies the unit-invariance property (8). Then, up to a strictly positive multiplicative constant, $f(\alpha)$ must take the form*

$$f(\alpha) = \left[\alpha^{\gamma}(1-\alpha)^{1-\gamma}\right]^{\frac{\sigma}{1-\sigma}}. \tag{9}$$

Proof. See appendix. ■

It is difficult to give any economically intuitive interpretation for any function form for f in (9) for which $\gamma \notin (0,1)$. Regardless of the elasticity of substitution, if $\gamma \notin (0,1)$, the firm will employ only one factor of production. The "efficiency enhancing" interpretation of f made above requires restricting our attention to the set of unit-invariant functional forms with $\gamma \in (0,1)$. This, however, gives us exactly what is required. Since, by assumption $\sigma < 1$, f then satisfies $f(0) = f(1) = 0$. So the assumption $\sigma < 1$ is important for two reasons: it maintains the strict essentiality of the production function for $\alpha \in (0,1)$, and it means that whenever we impose a unit-invariant functional form for f to which we can give a meaningful economic interpretation, that function must satisfy $f(0) = f(1) = 0$. Hence the strict essentiality of the production process is ensured. In the case $\sigma > 1$ we will have an asymmetric equilibrium where firms specialize in either factor. Finally, note that with these assumptions $f(\alpha)$ is maximized at $\alpha = \gamma$; γ will turn out to be the long-run capital share.

Since, given (9), the change of units leaves $f(.)$ invariant, and since σ is invariant to a change of units, then so is γ. Thus we are excused from any of the normalization considerations that often surround CES; a change in units only produces a change in the efficiency parameter Γ which from (8) is:[12]

$$\Gamma' = \phi(s)\Gamma = s^{\frac{\sigma(1-\gamma)}{\sigma-1}}\Gamma. \tag{10}$$

2.4 Dynamics

Using (9) we can now write down the firm's problem. This takes on a particularly simple form. Setting $\theta = (1-\alpha)/\alpha$, we can write

$$Y = \Gamma\left(\theta^{\gamma-1}(BK)^{\frac{\sigma-1}{\sigma}} + \theta^{\gamma}(AL)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \tag{11}$$

If $r+\delta$ and w are respectively the rental rates for capital and labor, the first order conditions with respect to K and L are:

$$(\Gamma B)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y}{K}\right)^{\frac{1}{\sigma}}\theta^{\gamma-1} = r+\delta. \tag{12}$$

$$(\Gamma A)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y}{L}\right)^{\frac{1}{\sigma}}\theta^{\gamma} = w. \tag{13}$$

As usual, $Y = (r+\delta)K + wL$. Holding θ constant, the elasticity of substitution is σ. However, of course, θ is not constant and in fact we can straightforwardly see that the elasticity of substitution between the two factors is unity. The first order condition for θ is

$$\frac{\sigma}{\sigma-1}\Gamma^{\frac{\sigma-1}{\sigma}}Y^{\frac{1}{\sigma}}\left((\gamma-1)\theta^{\gamma-2}(BK)^{\frac{\sigma-1}{\sigma}} + \gamma\theta^{\gamma-1}(AL)^{\frac{\sigma-1}{\sigma}}\right) = 0, \tag{14}$$

or equivalently,

$$\theta = \frac{1-\gamma}{\gamma}\left(\frac{BK}{AL}\right)^{\frac{\sigma-1}{\sigma}}. \tag{15}$$

Substituting (15) in (12) and (13) immediately implies a unitary elasticity of substitution between the two factors. Using the envelope theorem, the required second order condition for (14) is

[12] Note again the marginal product of capital, which is $\alpha\left(\Gamma f(\alpha)\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{Y}{B^{1-\sigma}K}\right)^{\frac{1}{\sigma}} = \Gamma^{\frac{\sigma-1}{\sigma}}\left(\frac{\alpha}{1-\alpha}\right)^{1-\gamma}\left(\frac{Y}{B^{1-\sigma}K}\right)^{\frac{1}{\sigma}}$. We can then easily see that the marginal product of capital is unaffected by a change in the units for labor since, from (10), $\Gamma^{\frac{\sigma-1}{\sigma}}\left(\frac{\alpha}{1-\alpha}\right)^{1-\gamma} = \Gamma'^{\frac{\sigma-1}{\sigma}}\left(\frac{\alpha'}{1-\alpha'}\right)^{1-\gamma}$.

$$\frac{\sigma}{\sigma-1}\Gamma^{\frac{\sigma-1}{\sigma}}Y^{\frac{1}{\sigma}}\gamma\theta^{\gamma-2}(AL)^{\frac{\sigma-1}{\sigma}}<0. \tag{16}$$

This is always satisfied for $\sigma<1$. We can then rely on (16) and the strict essentiality of the production function for a symmetric solution.[13] We now in fact have a Cobb-Douglas production function with an exponent γ on capital. Substituting (12) and (13) into (14) then gives the capital share

$$\frac{(r+\delta)K}{Y}=\gamma$$

and (15) into (11) gives

$$Y=\Gamma\left(\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right)^{\frac{\sigma}{1-\sigma}}(BK)^{\gamma}(AL)^{1-\gamma}. \tag{17}$$

This might appear a rather troublesome way to simply obtain a Cobb-Douglas production function. However, of course, all we now need is to introduce some dynamics with adjustment costs in θ to produce a system where the elasticity of substitution between factors falls short of one in the short run.[14] As these adjustments costs become large, the short-run elasticity of substitution between capital and labor will approach σ. Hence, we can have short-run dynamics with gross factor complementarity but balanced growth in the long-run regardless of whether productivity growth is labor-augmenting or capital-augmenting.[15]

In the absence of adjustment costs, since the above gives us Cobb-Douglas, we can see from (15) that with purely labor augmenting technical progress, θ will tend to a finite and positive steady-state.[16] If there is capital augmenting technical progress such that $\Gamma B\longrightarrow\infty$ in the long-run, then θ must also tend to zero in the long-run remembering that $\sigma<1$. We can deduce the long-run growth rates under balanced growth from (12) and (13) as follows, writing g_X as the long-run growth rate of X_t :

$$g_{\theta}=-\frac{1}{1-\gamma}\left(\frac{1-\sigma}{\sigma}\right)(g_{\Gamma}+g_B)<0 \tag{18}$$

[13] This can also be thought of as following from having a well-defined second order condition.

[14] One might also want to consider the entry of new firms. If one assumes that a new firm entering the market faces the same adjustment costs – for instance if θ_{t-1} represents the 'standard blueprint' in $t-1$ and each firm, new or old, faces an adjustment cost in departing from this – then adjustment costs in θ are enough. If, however, a new firm can make any choice of θ_t then we might also need adjustments costs in either K or L given that one might more naturally assume that the firm starts out with $K=L=0$. Otherwise each period would be populated only by new firms at the optimal level of θ.

[15] It is worth insisting on the fact that, in the long-run BGP, both kinds of technical progress are undistinguishable, but they generate very different short-run responses.

[16] Since $\frac{K}{B^{\gamma/(1-\gamma)}AL}$ tends to a finite (strictly) positive value in a neo-classical growth setting.

$$g_Y = g_A + g_L + \frac{1}{1-\gamma} g_\Gamma + \frac{\gamma}{1-\gamma} g_B \tag{19}$$

noting that (19) is exactly as we would expect from Cobb-Douglas.

Since adjustment costs alter the short-run dynamics rather than the long run growth path, neither of these conclusions is changed when they are introduced. Adjustment costs should be specified in terms of θ rather than α. This is important since the ratio θ_t/θ_{t-1} is invariant to the choice of units whereas α_t/α_{t-1} is not. Ideally, of course, the dynamics should be embedded in the appropriate general equilibrium framework for the question the researcher wishes to address. However, we provide a very simple example for a representative firm, treating factor prices as exogenous and the price of output as constant. Suppose the costs of adjusting θ are proportional to output and are denoted by $\varphi\left(\frac{\theta_t}{\theta_{t-1}}\right) Y$. The firm's problem is then to maximize

$$\sum_{t=0}^{\infty} \left\{ \left[\prod_{s=0}^{t} \left(\frac{1}{1+r_t} \right) \right] \left[Y_t \left\{ 1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}} \right) \right\} - (r_t + \delta) K_t - w_t L_t \right] \right\} \tag{20}$$

where $\varphi(1) = 0$, $\phi \geqslant 0$, $\phi'' > 0$ and

$$Y_t = \Gamma_t \left(\theta_t^{\gamma-1} (B_t K_t)^{\frac{\sigma-1}{\sigma}} + \theta_t^{\gamma} (A_t L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{21}$$

The first order conditions are then:

$$\left\{ 1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}} \right) \right\} (\Gamma_t B_t)^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{K_t} \right)^{\frac{1}{\sigma}} \theta_t^{\gamma-1} = r_t + \delta \tag{22}$$

$$\left\{ 1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}} \right) \right\} (\Gamma_t A_t)^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{L_t} \right)^{\frac{1}{\sigma}} \theta_t^{\gamma} = w_t \tag{23}$$

$$\begin{aligned} &\frac{\sigma}{\sigma-1} \left[\gamma \left\{ 1 - \varphi\left(\frac{\theta_t}{\theta_{t-1}} \right) \right\} - \frac{(r_t+\delta)K_t}{Y_t} \right] - \\ &- \left\{ \frac{\theta_t}{\theta_{t-1}} \varphi'\left(\frac{\theta_t}{\theta_{t-1}} \right) - \frac{1}{1+r_t} \frac{\theta_{t+1}}{\theta_t} \phi'\left(\frac{\theta_{t+1}}{\theta_t} \right) \frac{Y_{t+1}}{Y_t} \right\} = 0 \end{aligned} \tag{24}$$

Equations (22) to (24) provide a system that can be readily incorporated in many macroeconomic models. Once one has decided on the stochastic processes governing the evolution of technology, it can be straightforwardly written in 'intensive' form using the growth rates given in (18) and (19). Note, again, that the special case of pure Cobb-Douglas is achieved by setting adjustment costs $\tau = 0$

rather than setting $\sigma = 1$ in (21). In (24), as θ asymptotes to a long-run value, strictly positive in the presence of bounded capital-augmenting or purely labor-augmenting technical progress, we can see that the capital share will tend towards γ ensuring balanced growth. If adjustment costs are large, then θ_t will have a sluggish response to short-run changes in factor prices, and so from (22) and (23) the elasticity of substitution will be close to σ.

Interestingly, with exponential capital-augmenting growth we have a balanced growth path with a constant capital share in steady state but this might not strictly be γ; the *growth rate* of capital-augmenting efficiency can affect the *level* of capital share. This is because if capital-augmenting efficiency grows exponentially, there is continual adjustment towards Cobb-Douglas without ever quite reaching it.[17] Note, however, if adjustment costs are differentiable, so $\phi'(1) = 0$, one can easily show that the departure of the capital share from γ is second order in the growth and discount rates.[18]

3 Conclusions

We provide an easily implementable solution to the problems that arise as a result of the Steady State Growth Theorem in reconciling short-run dynamics with long run stylized facts regarding balanced growth. The intuition for the mechanism given here is as follows: firms can choose production techniques via the distribution parameter of the production function, but this choice is constrained in two dimensions. The first constraint arises through considering efficiency; extreme choices of technique are punished since all tasks cannot be performed by either capital or labor. Secondly, the choice of technique is constrained through time by adjustment costs. In the long run, continued adjustment towards the optimal (adjustment-cost-free) choice leads to a Cobb-Douglas production, but in the short-run the partial adjustment results in an elasticity of substitution that may be much lower than one.

This produces a class of production functions that results in a long-run unitary elasticity of substitution but display a short-run elasticity of substitution less than one. As such, the approach presented is very general and, importantly, tractable and implementable within a wide variety of macroeconomic models without requiring explicit models of R&D. It also allows for models where technical progress is not restricted to the labor-augmenting case, and hence with richer short- to

[17] We could draw an analogy with a train that has a stretch in the couplings between its carriages; as the speed of the train increases the carriages move at the same speed but the distance between carriages increases as the coupling stretches.

[18] With σ approaching zero (Leontief case) or kinked adjustment costs, there are therefore some potentially interesting results linking the long-run capital share to technological growth rates.

medium-run dynamics for factor shares, relative factor prices, and the labor wedge. Extensions of this approach for further research can consider its introduction in general equilibrium business cycle models, as well as the consideration of multi-sector models.

APPENDIX

A Proof of Lemma 2

Suppose for any α and s on the intervals $\alpha \in [0,1]$, and $s \in (0,\infty)$, there exists a function $\phi(s)$ independent of α such that a continuously differentiable function $f(\alpha)$ satisfies

$$\frac{f(\alpha)}{(\alpha+(1-\alpha)s)^{\frac{\sigma}{1-\sigma}}} = f\left(\frac{\alpha}{\alpha+(1-\alpha)s}\right)\phi(s). \tag{A.1}$$

Clearly $\phi(1)$ must equal 1. On the interval $\alpha \in (0,1]$, we can then define a continuously differentiable function $g(.)$ such that

$$g(\alpha) \equiv \frac{f(\alpha)}{\alpha^{\frac{\sigma}{1-\sigma}}} \tag{A.2}$$

in which case from (A.1) we have

$$\frac{g\left(\frac{\alpha}{\alpha+(1-\alpha)s}\right)}{g(\alpha)} = \frac{1}{\phi(s)}. \tag{A.3}$$

Now set $s = 1+\varepsilon$ where ε is small, so

$$\frac{\alpha}{\alpha+(1-\alpha)s} = \alpha + \alpha(1-\alpha)\varepsilon + o(\varepsilon^2),$$

and consider the Taylor expansion in ε of the left hand side of (A.3) around 1:

$$1 + \alpha(1-\alpha)\frac{g'(\alpha)}{g(\alpha)}\varepsilon + o(\varepsilon^2) = \frac{1}{\phi(1+\varepsilon)}. \tag{A.4}$$

Since $\phi(1) = 1$, the Taylor expansion of $1/\phi(1+\varepsilon)$ around 1 must take the form $1 - K_1\varepsilon - K_2\varepsilon^2 + ..$ noting crucially that since ϕ is independent of α, then K_1, K_2 etc. must also be. Hence equating terms of order ε, we have:

$$\alpha(1-\alpha)\frac{g'(\alpha)}{g(\alpha)} = -K_1. \tag{A.5}$$

Let $\theta \equiv \frac{1-\alpha}{\alpha}$, and define another continuously differentiable function $h(.)$ such that

$$g(\alpha) \equiv h(\theta). \tag{A.6}$$

Then

$$\alpha^2 g'(\alpha) = -h'(\theta)$$

and (A.5) becomes

$$\frac{h'(\theta)}{h(\theta)} = \frac{K_1}{\theta}. \tag{A.7}$$

But then the solution to the differentiable equation (A.6) must take the form

$$h(\theta) = C\theta^{K_1} \tag{A.8}$$

for some constant C. Hence, using equations (A.2) and (A.6), $f(.)$ must take the following form, up to a multiplicative constant, on the interval $(0, 1]$ and therefore by continuity on $[0, 1]$:

$$f(\alpha) = \left[\alpha^{\gamma}(1-\alpha)^{1-\gamma}\right]^{\frac{\sigma}{1-\sigma}} \tag{A.9}$$

for some real γ. This gives necessity, and for sufficiency it can be easily shown that (A.9) satisfies the unit-invariance condition (A.1). Note that this also implies that all the further terms in the Taylor expansion on the left-hand side of (A.4) do not vary with α, and this indeed can be verified directly and straightforwardly by continuing the Taylor expansion to higher order terms and substituting in the expression for $g(\alpha)$ that comes from (A.8).

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