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Unitary Elasticity of Substitution**

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Abstract

We present a simple production technology in which the choice of production technique results in a balanced growth path even in the presence of capital-augmenting technical progress. Given a particular choice of technique, the production function is CES with a less than unitary elasticity of factor substitution. The form of this production technology is also invariant to the choice of units, allowing us to abstract from the normalization considerations that often accompany the use of CES. The approach yields a balanced growth path but short-run time-varying factor shares without requiring an explicit model of the R&D sector.

JEL Classification: E25, O33, O40.

Keywords: Balanced growth, capital-augmenting technical progress, measurement units, elasticity of substitution.

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1 Introduction

A widely used result in neoclassical growth theory is that, for a balanced growth path (BGP) to exist, technical progress must be labor-augmenting or the production function must be Cobb-Douglas (Uzawa, 1961).¹ An important question arising from this theorem is then “what mechanisms ensure that technical progress is labor-augmenting?” The early literature on induced-innovation by Kennedy (1964), Samuelson (1965), and Drandakis and Phelps (1966), inspired by Hicks (1932), viewed this as the result of firms introducing innovations that save on expensive factors in the face of changes in relative factor prices. More recently, this line of thought has been revisited by Acemoglu (2002, 2003, 2007), Zeira (1998), and Zuleta (2008), amongst others. Jones (2005), on the other hand, focuses on the shape of the production function. He stresses that even if the short-run (local) production function is of the “constant elasticity of substitution” (CES) type with a less than unitary elasticity, if techniques (ideas) follow a Pareto distribution, the long-run (global) production function will be Cobb-Douglas and all technical progress labor-augmenting.

The question we pose in this paper is: “can we obtain a BGP in the presence of capital-augmenting technical progress through the imposition of a simple CES-type production technology that allows an optimal choice of technique?” In many macroeconomic applications – particularly where innovation itself is not the main question of interest – such an approach might prove attractive. The main advantage in such cases would to give the researcher the ability to include capital-augmenting technical progress and a less than unitary elasticity of substitution simply via a suitable specification of the production technology, without the usual consequent worry of the lack of a BGP. Hence this would give the researcher a straightforward way to depart from the standard Cobb-Douglas assumption, which sits at odds with the observed large (short to medium run) variations in factor shares.

The first requirement of such an approach is that it must be based on an optimizing choice of production technique. We use the choice of technique here specifically to refer to the relative reliance on capital or labor in production, i.e. the choice of α in equation (1). The logic of the paper is as follows. Firstly, we argue that, with a standard CES specification, simply choosing the optimal technique as well as the optimal quantities of labor and capital is unlikely to provide a positive answer to the question of balanced growth. The main issue is that the outcome is asymmetric and involves firms specializing only in capital or only in labor. We avoid this problem by multiplicatively augmenting the CES with a function $f(\alpha)$ that penalizes extreme choices of α , while leaving us with a production function

¹ See also Jones and Scrimgeour (2008) for a proof.

that remains CES when the choice of technique α is given. While this may appear ad-hoc, it captures the economically intuitive idea that while firms may alter their relative reliance on capital or labor, extreme choices such as relying *exclusively* on capital are likely to prove relatively unproductive. Furthermore, the natural worry that choice of $f(\alpha)$ is likely to be ad-hoc is mitigated when we consider the impact of a change of units. In fact, there is effectively only one specification for $f(\alpha)$ that gives our problem of making an optimal choice of production technique a form that is satisfactorily invariant to an arbitrary change of measurement units.

Using this specification for $f(\alpha)$ then leads us to a production technology that both allows us to abstract from considerations of normalization that often arise with the use of CES (see La Grandville 1989, Klump and La Grandville 2000, and León-Ledesma *et al* 2010), but more importantly ensures balanced growth in the long-run even with capital-augmenting technical progress. A less than unitary elasticity of substitution in the short-run dynamics is then simply obtained by imposing adjustment costs on the choice of technique since the production technology *is* CES when the technique is held constant. Our optimal choice of capital intensity then generates a production function whose short run elasticity of substitution is below one but equals unity in the long-run. That is, the simple approach we outline here is capable of generating the short-run *vs.* long-run result in Jones (2005) by using a model of endogenous choice of technique. The approach is also very general, and can be used in exogenous savings and optimal growth model settings equally.

The relevance of our approach is worth emphasizing. The early models of “induced innovation” relied on assumptions about firm behavior that were not well founded on microeconomic principles. As stressed by Acemoglu (2001), the usual assumption in these models is that firms maximized the rate of growth of output (the rate of cost reduction), simply because aggregate technology showed increasing returns. Hence, “to go beyond the heuristics of maximizing the instantaneous rate of cost reduction, we need a micro-founded model of innovation” (Acemoglu, 2001, p. 470). Our approach presents a way to resolve these issues preserving the profit maximization behavior of firms without requiring explicit models of innovation. While modeling innovation is of course a fundamental issue, its complexity may represent an obstacle to its introduction in macroeconomic models preserving BGP when the research question does not concern innovation itself, making departures from a standard Cobb-Douglas framework difficult. In models that are focussed on innovation, this approach also liberates the researcher from the constraint that the innovation process itself should ensure balanced growth in the long run, since the choice of technique provides an alternative mechanism by which this occurs.

2 The production technology

Take the CES production, omitting any time subscripts:

$$Y = \Gamma \left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (1)$$

where K is capital, L labor, B and A are capital- and labor-augmenting technical progress functions, Γ is a neutral efficiency parameter, σ is the elasticity of substitution between K and L , and α is the capital intensity of production. As is well known, of the three parameters α , Γ , and σ , only σ is invariant to the choice of units. For example, suppose we maintain the same units for Y and K but use a different measure of labor $L' = m^{\frac{\sigma}{1-\sigma}} L$ where m is constant. Then equation (1) becomes:

$$Y = \Gamma' \left(\alpha' (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha')(AL')^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (2)$$

where $\alpha' = \frac{\alpha}{\alpha+(1-\alpha)m}$ and $\Gamma' = \Gamma (\alpha + (1-\alpha)m)^{\frac{\sigma-1}{\sigma}}$.

Suppose now firms are faced with (1) and factor prices $r + \delta$ and w , but now have the additional choice of choosing α as well as the inputs K and L , and we have a standard free entry condition. Clearly from (1) we will not have a satisfactory second order condition for α ; we are in fact about to deem the outcome of this problem (described in lemma 1) as ‘unrealistic’ and reformulate it. However it is useful for further discussion and the interesting point is that even though all firms are identical, the outcome must be asymmetric.

Lemma 1 *Some firms will only employ capital and others will employ only labor. No firm will employ both.*

Proof. Suppose $r + \delta < \Gamma B$. Then a firm entering the market choosing $\alpha = 1$ will make a strictly positive profit $(\Gamma B - r - \delta)K$ violating the free-entry condition. Similarly if $w < \Gamma A$ firms can enter and make strictly positive profits choosing $\alpha = 0$. Thus in equilibrium the factor prices must be $r + \delta = \Gamma B$ and $w = \Gamma A$. Thus a firm choosing $\alpha \in (0, 1)$ will earn a profit of Γ times

$$\left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - BK - AL.$$

But $(\alpha (BK)^{(\sigma-1)/\sigma} + (1-\alpha)(AL)^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)} \leq \max(BK, AL)$ and if both K and L are strictly positive, $\max(BK, AL) < BK + AL$. Hence no firm that employs both factors can make positive profits. ■

Much of the paper is concerned with how to modify the problem of choosing α in order to end up with a more sensible solution. This modification turns out

to produce a very simple framework that also has some very convenient properties for balanced growth. We make two changes. These are partly aimed in the first instance at ensuring strict essentiality of the production function in order to rule out the type of asymmetric equilibrium of lemma 1, and ensuring that an appropriate second order condition is satisfied for α . The CES production function does not satisfy strict essentiality when $\sigma > 1$. This does not normally matter since the marginal product of each factor tends to infinity at zero, but when α is a choice variable the possibility of one firm specializing in one factor and bidding up the price of that factor becomes a real one, which can mean an asymmetric equilibrium is possible.²

The first change is therefore to impose $\sigma < 1$, though this only ensures strict essentiality for CES if $\alpha \in (0, 1)$. The second change is to introduce a term $f(\alpha)$ into the production function as shown in (3):

$$Y = \Gamma f(\alpha) \left(\alpha (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (3)$$

The term $f(\alpha)$ essentially punishes extreme choices of α , so while the firm can choose α , we are assuming that the firm cannot produce output with only labor or only capital. Therefore we impose $f(0) = f(1) = 0$ with $f(\alpha)$ strictly positive on $\alpha \in (0, 1)$. The choice of α is also a continuous one so $f(\alpha)$ is continuously differentiable and positive on $\alpha \in [0, 1]$. $f(\alpha)$ provides a time-invariant technological constraint on the choice of technique.

In light of this, we face the requirement to make a consistent representation of the production function in the face of a change of units. Given the introduction of $f(\alpha)$, this is perhaps the key insight of our approach. It leads us to impose a very specific functional form on $f(\alpha)$ that gives us our main result. Suppose we consider the same change of units above where we use a different measure of labor $L' = m^{\frac{\sigma}{1-\sigma}} L$ where m is constant. We wish to write:

$$Y = \Gamma' \left[f(\alpha') \left(\alpha' (BK)^{\frac{\sigma-1}{\sigma}} + (1-\alpha')(AL')^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right], \quad (4)$$

where, as before, we must have $\alpha' = \frac{\alpha}{\alpha + (1-\alpha)m}$.

If (3) is to be a consistent representation of the production function in the face of a change of units, $f(\cdot)$ *must* capture the *whole* of the dependency of Y on α' outside the standard CES term in (4); so in (4) we do not want any term in α' outside the square brackets. Equivalently, we require the functional form for $f(\cdot)$

² One could fairly dismiss this as an unrealistic ‘nuisance’ equilibrium. However in situations where the elasticity of substitution between two factors might be above 1, skilled and unskilled labor for example, allowing firms to choose might explain specialisation among firms in one factor or the other without relying on any ex-ante heterogeneity.

to stay invariant following the change of units. Hence Γ' cannot be a function of α' (or therefore α) and can only depend on Γ , m , and σ . Suppose $\Gamma' = g(m; \sigma)\Gamma$ for some function $g(m; \sigma)$. We must then have:

$$\frac{f(\alpha)}{(\alpha + (1 - \alpha)m)^{\frac{\sigma}{1-\sigma}}} = f(\alpha')g(m) = f\left(\frac{\alpha}{\alpha + (1 - \alpha)m}\right)g(m) \quad (5)$$

If $f(\alpha) = \alpha^a(1 - \alpha)^b$ where $a > 0$ and $b > 0$ so that $f(0) = f(1) = 0$, then (5) simply requires $a + b = \sigma/(1 - \sigma)$. So the assumption $\sigma < 1$ buys us two things: it maintains the strict essentiality of the production function for $\alpha \in (0, 1)$, and it allows us to impose a unit-invariant functional form for f that satisfies $f(0) = f(1) = 0$. In the case $\sigma > 1$ we will have an asymmetric equilibrium where firms specialize in either factor. We can then write

$$f(\alpha) = [\alpha^\gamma(1 - \alpha)^{1-\gamma}]^{\frac{\sigma}{1-\sigma}}, \quad (6)$$

where $\gamma \in (0, 1)$, noting that $f(\alpha)$ is maximized at $\alpha = \gamma$. Since, given (6), the change of units leaves $f(\cdot)$ invariant, and since σ is invariant to a change of units, then so is γ . Thus we are excused from any of the normalization considerations that often surround CES; a change in units only produces a change in the efficiency parameter Γ with

$$\Gamma' = g(m)\Gamma = m^{\frac{\sigma(1-\gamma)}{1-\sigma}}\Gamma.$$

Using (6) we can now write down the firm's problem. This takes on a particularly simple form. Setting $\theta = (1 - \alpha)/\alpha$, we can write

$$Y = \Gamma \left(\theta^{\gamma-1}(BK)^{\frac{\sigma-1}{\sigma}} + \theta^\gamma(AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \quad (7)$$

If $r + \delta$ and w are respectively the rental rates for capital and labor, the first order conditions with respect to K and L are:

$$\Gamma^{\frac{\sigma-1}{\sigma}} \left(\frac{Y}{BK} \right)^{\frac{1}{\sigma}} B\theta^{\gamma-1} = r + \delta. \quad (8)$$

$$\Gamma^{\frac{\sigma-1}{\sigma}} \left(\frac{Y}{AL} \right)^{\frac{1}{\sigma}} A\theta^\gamma = w. \quad (9)$$

As usual, $Y = (r + \delta)K + wL$. Holding θ constant, the elasticity of substitution is σ . However, of course, θ is not constant and in fact we can straightforwardly see that the elasticity of substitution between the two factors is unity. The first order condition for θ is

$$\frac{\sigma}{\sigma-1} \Gamma^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} \left((\gamma-1) \theta^{\gamma-2} (BK)^{\frac{\sigma-1}{\sigma}} + \gamma \theta^{\gamma-1} (AL)^{\frac{\sigma-1}{\sigma}} \right) = 0, \quad (10)$$

or equivalently,

$$\theta = \frac{1-\gamma}{\gamma} \left(\frac{BK}{AL} \right)^{\frac{\sigma-1}{\sigma}}. \quad (11)$$

Substituting (11) in (8) and (9) immediately implies a unitary elasticity of substitution between the two factors. Using the envelope theorem, the required second order condition for (10) is

$$\frac{\sigma}{\sigma-1} \Gamma^{\frac{\sigma-1}{\sigma}} Y^{\frac{1}{\sigma}} \gamma \theta^{\gamma-2} (AL)^{\frac{\sigma-1}{\sigma}} < 0. \quad (12)$$

This is always satisfied for $\sigma < 1$. We can then rely on (12) and the strict essentiality of the production function for a symmetric solution.³ Substituting (8) and (9) into (10) then gives the capital share

$$\frac{(r+\delta)K}{Y} = \gamma.$$

We then effectively have the Cobb-Douglas production function with an exponent γ on capital. This might appear a rather troublesome way to simply obtain a Cobb-Douglas production function. However, of course, all we now need is to introduce some dynamics with adjustment costs in θ to produce a system where the elasticity of substitution between factors falls short of one in the short run.⁴ As these adjustments costs become large, the short-run elasticity of substitution between capital and labor will approach σ . Hence, we can have short-run dynamics with a less than unitary elasticity of substitution but balanced growth in the long run regardless of whether productivity growth is labor-augmenting or capital-augmenting.

Before introducing adjustments costs, it is worth noting the long-run limiting properties of θ in a standard neo-classical growth context in their absence. Since

³This can really also be thought of as following from having a well-defined second order condition.

⁴One might also want to consider the entry of new firms. If one assumes that a new firm entering the market faces the same adjustment costs – for instance if θ_{t-1} represents the ‘standard blueprint’ in $t-1$ and each firm, new or old, faces an adjustment cost in departing from this, then adjustment costs in are enough. If, however, a new firm can make any choice of θ_t then we might also need adjustments costs in either K or L given that one might more naturally assume that the firm starts out with $K=L=0$. Otherwise each period would be populated only by new firms at the optimal level of θ .

the above gives us Cobb-Douglas, we can see from (11) that in the presence of purely labor augmenting technical progress, θ will tend to a finite and positive steady-state.⁵ If there is any capital augmenting technical progress, θ must tend to zero in the long-run, remembering that $\sigma < 1$. Since adjustment costs alter the short-run dynamics rather than the long run steady state, neither of these conclusions is changed when they are introduced.

Adjustment costs should be specified in terms of θ rather than α . This is important since the ratio θ_t/θ_{t-1} is invariant to the choice of units whereas α_t/α_{t-1} is not. Ideally, of course, the dynamics should be embedded in the appropriate general equilibrium framework for the question the researcher wishes to address. However, we provide a very simple partial example for a representative firm, treating factor prices as exogenous and the price of output as constant. Suppose the costs of adjusting θ , denoted by φY say, are proportional to output. The firm's problem is then to maximize

$$\Pi = \sum_{t=0}^{\infty} \beta^t \left\{ Y_t \left[1 - \varphi \left(\frac{\theta_t}{\theta_{t-1}} \right) \right] - (r_t + \delta)K_t - w_t L_t \right\} \quad (13)$$

where say $\varphi(x) = 1 - e^{-\frac{1}{2}\tau(x-1)^2}$ and

$$Y = \Gamma \left(\theta^{\gamma-1} (BK)^{\frac{\sigma-1}{\sigma}} + \theta^{\gamma} (AL)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \quad (14)$$

The first order conditions are then:

$$e^{-\frac{1}{2}\tau\left(\frac{\theta_t-\theta_{t-1}}{\theta_{t-1}}\right)^2} \Gamma_t^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{B_t K_t} \right)^{\frac{1}{\sigma}} B_t \theta_t^{1-\gamma} = r_t + \delta \quad (15)$$

$$e^{-\frac{1}{2}\tau\left(\frac{\theta_t-\theta_{t-1}}{\theta_{t-1}}\right)^2} \Gamma_t^{\frac{\sigma-1}{\sigma}} \left(\frac{Y_t}{A_t L_t} \right)^{\frac{1}{\sigma}} A_t \theta_t^{-\gamma} = w_t \quad (16)$$

$$\frac{\sigma}{\sigma-1} \left(\gamma - \frac{(r_t + \delta)K_t}{Y_t} \right) - \tau \left\{ \frac{\theta_t}{\theta_{t-1}} \left(\frac{\theta_t - \theta_{t-1}}{\theta_{t-1}} \right) - \beta \frac{\theta_{t+1}}{\theta_t} \left(\frac{\theta_{t+1} - \theta_t}{\theta_t} \right) \frac{Y_{t+1}}{Y_t} \right\} = 0 \quad (17)$$

Equations (15) to (17) provide a system that can be readily incorporated in many macroeconomic models. Note again that the special case of pure Cobb-Douglas is achieved by setting adjustment costs $\tau = 0$ rather than setting $\sigma = 1$ in (14). In (17), as θ tends to a steady-state value, strictly positive and finite in the presence of purely labor augmenting technical progress or zero otherwise, we can see that the capital share will tend towards γ ensuring balanced growth. If τ and

⁵ Since tends to a finite (strictly) positive value in a neo-classical growth setting.

so adjustment costs are large, then θ_t will have a sluggish response to short-run changes in factor prices, and so from (15) and (16) the elasticity of substitution will be close to σ .

3 Conclusions

Standard neoclassical growth models require that technical progress be labor-augmenting or the production function Cobb-Douglas for a balanced growth path (BGP) to exist. Considerable effort has been placed in explaining why technical progress is labor-augmenting or the shape of the production function is Cobb-Douglas (i.e. unit elasticity of substitution) in the long run. We ask the question of whether a BGP can be obtained with capital-augmenting technical progress and a less than unitary elasticity of substitution, while preserving the profit optimizing nature of firms. We show that this is possible in a context where firms optimally choose their capital intensity in production, which we have labeled here as choice of production technique.

Much of the paper is concerned with the most appropriate mathematical representation of this choice. Simply choosing the optimal technique and quantities of labor and capital in a standard CES specification leads to an unlikely outcome which is asymmetric and involves firms specializing only in capital or only in labor. We provide a simple representation that maintains a constant elasticity of substitution between capital and labor when the choice of technique is given, and require that it satisfies two properties (i) strict essentiality of the production process even when there is a free choice of technique (ii) it should take a form that is invariant to a change of units. A nice feature of the analysis is that effectively the only straightforward representation of the choice of technique that satisfies these two properties also produces a solution to the problem of balanced growth described above. The approach is therefore very general and can potentially be used in a wide variety of macroeconomic models with consequences for the dynamics of factor shares, relative factor prices, and the labor wedge.

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